TAU-FUNCTION OF DISCRETE ISOMONODROMY TRANSFORMATIONS AND PROBABILITY

D. ARINKIN, A. BORODIN

Abstract. We introduce the τ-function of a rational d-connection and its isomonodromy transformations. We show that in a continuous limit our τ-function agrees with the Jimbo-Miwa-Ueno τ-function, compute the τ-function for the isomonodromy transformations leading to difference Painlevé V and difference Painlevé VI equations, and prove that the gap probability for a wide class of discrete random matrix type models can be viewed as the τ-function for an associated d-connection.

Introduction

The theory of isomonodromy deformations of rational connections over $\mathbb{P}^1$ has a long history. It was pioneered in the beginning of the twentieth century by R. Fuchs and L. Schlesinger, and after being dormant for fifty years, it sprang back to life with the work of M. Jimbo, T. Miwa, Y. Môri, M. Sato, T. Ueno and other members of the famous Kyoto school in the late seventies. Since then the theory found a number of applications in statistical physics (see e.g. a series of papers on holonomic quantum fields by the Kyoto school), random matrix theory (see e.g. [19], [32], [29], [18], [8]), theory of Frobenius manifolds (see [11]), and representation theory (see [8]).

A central role in the theory of isomonodromy deformations is played by the so-called τ-*function* — a holomorphic function on the universal covering space of the space of parameters of the connection, which vanishes when the corresponding isomonodromy deformation fails to exist. The isomonodromy τ-function was initially introduced and studied by M. Jimbo, T. Miwa, and T. Ueno in [20], [21], [22]. It found various interpretations in applications; e.g. in random matrix theory the τ-function appears as the *gap probability* — the probability that no eigenvalues of the random matrix are present in a union of intervals. This fact can be seen as one reason why the gap probabilities for one-interval gaps are often expressible through solutions of the classical Painlevé equations, see e.g. [19], [28], [32], [1], [8], [18], [12]-[16] for details.

The theory of isomonodromy transformations of *difference* rational connections (d-connections, for short) on $\mathbb{P}^1$ is much younger. It was suggested by one of the authors in [6] and employed in [5], [7], [26], [3], [31]. There are presently two principal applications of the theory: On the one hand, isomonodromy transformations of d-connections with few singularities provide a key for understanding the geometry of discrete Painlevé equations from Sakai's hierarchy (see [30] for the hierarchy and [3], [31] for explicit connections). On the other hand, the discrete isomonodromy

Date: April 17, 2007.

transformations can be used to compute the gap probabilities in various discrete probabilistic models of random matrix type, see [5], [7].

The main goal of this paper is to introduce the notion of the τ-function of a rational d-connection and its isomonodromy transformations. We also show that in a continuous limit our τ-function agrees with the conventional one; we compute the τ-function for the isomonodromy transformations leading to difference Painlevé V (dPV) and difference Painlevé VI ($dPVI$) equations (in the terminology of [3]), and we prove that the gap probability for a wide class of discrete random matrix type models can be viewed as the τ-function for an associated d-connection.

Let us describe our results in more detail.

Let $\mathcal{L}$ be a vector bundle on $\mathbb{P}^{\mathbf{1}}$ of rank m. Define a one-dimensional vector space $\det\mathrm{R}\Gamma(\mathcal{L})$ by

$$\det\mathrm{R}\Gamma(\mathcal{L}) = \det(H^0(\mathbb{P}^{\mathbf{1}},\mathcal{L})) \otimes (\det(H^1(\mathbb{P}^{\mathbf{1}},\mathcal{L})))^{-1}.$$

Recall that $H^0(\mathbb{P}^{\mathbf{1}},\mathcal{L})$ is the space of global regular sections of $\mathcal{L}$, and $H^1(\mathbb{P}^{\mathbf{1}},\mathcal{L})$ can be interpreted as the space of obstructions for a Mittag-Leffler problem, see Section 1.4 for details. Both $H^0(\mathbb{P}^{\mathbf{1}},\mathcal{L})$ and $H^1(\mathbb{P}^{\mathbf{1}},\mathcal{L})$ are finite-dimensional.

In a sense, $\det\mathrm{R}\Gamma(\mathcal{L})$ is the only nontrivial way to associate to a vector bundle $\mathcal{L}$ a one-dimensional vector space. More precisely, we can view $\det\mathrm{R}\Gamma$ as a line bundle on the moduli space of vector bundles on $\mathbb{P}^{\mathbf{1}}$, and any other line bundle is its tensor power (see [27] and references therein for the statement and its generalizations).

The definition of $\det\mathrm{R}\Gamma$ makes sense (and is widely used) in a much more general situation; one description can be found in [25].

Observe that if $\mathcal{L} \simeq (\mathcal{O}(-1))^m$ then $H^0(\mathbb{P}^{\mathbf{1}},\mathcal{L}) = H^1(\mathbb{P}^{\mathbf{1}},\mathcal{L}) = 0$, so $\det\mathrm{R}\Gamma(\mathcal{L}) = \mathbb{C}$ and $\det\mathrm{R}\Gamma(\mathcal{L})^{-1} = \mathbb{C}$. In particular, there is a canonical element $1 \in \det\mathrm{R}\Gamma(\mathcal{L})^{-1}$.

Definition. Suppose $\mathcal{L}$ has slope -1; that is, $\deg(\mathcal{L}) = -m$. We define $\tau(\mathcal{L}) \in \det\mathrm{R}\Gamma(\mathcal{L})^{-1}$ by

$$\tau(\mathcal{L}) = \begin{cases} 1 & \text{if } \mathcal{L} \simeq (\mathcal{O}(-1))^m, \\ 0 & \text{otherwise.} \end{cases}$$

By itself, the element $\tau(\mathcal{L}) \in \det\mathrm{R}\Gamma(\mathcal{L})^{-1}$ provides almost no meaningful information. However, if $\mathcal{L}$ is equipped with an additional structure, the derivatives of τ might be meaningful. More precisely, given a d-connection on $\mathcal{L}$ (of a certain kind), we have a sequence of 'modifications' $\{\mathcal{L}_n\}_{n\in\mathbb{Z}}$ and a canonical isomorphism $\det\mathrm{R}\Gamma(\mathcal{L}_{n+1}) \stackrel{\sim}{\rightarrow} \det\mathrm{R}\Gamma(\mathcal{L}_n) \otimes S$, where $\mathcal{L}_n$ is a vector bundle on $\mathbb{P}^{\mathbf{1}}$, $\mathcal{L}_0 = \mathcal{L}$, and S is a one-dimensional vector space that does not depend on n. Therefore, the first ratio $\tau(\mathcal{L}_{n+1})/\tau(\mathcal{L}_n)$ is a functional on S, while the second ratio

$$\frac{\tau(\mathcal{L}_n)\tau(\mathcal{L}_{n+2})}{\tau^2(\mathcal{L}_{n+1})}$$

is a number (assuming that the denominator is nonzero).

All the modifications $\mathcal{L}_n$ are equipped with d-connections, which can be viewed as 'isomonodromy transformations' of the initial d-connection on $\mathcal{L}_0$. Explanations of the term can be found in [6], [26].

This paper is organized as follows.

Sections 1 contains general definitions.

In Section 2 we explicitly compute the ratios of the τ-function for modifications of three types: when two simple zeroes of $\mathcal{A}(z)$ shift in different directions, when a simple zero and a simple pole shift in the same direction, and when the shifting simple zero and simple pole coalesce. Note that these are the simplest modifications that preserve the degree of $\mathcal{L}$.

In Section 3 we consider a limit transition that turns a d-connection into an ordinary connection. We verify that the second difference logarithmic derivatives of our τ-function converge to the second logarithmic derivatives of the conventional isomonodromy τ-function for the limiting connection, see Theorem 3.1. It is worth pointing out that in the continuous situation the definition of the τ-function prescribes its first logarithmic derivatives rather than the second ones. However, in the difference situation the first derivatives are defined only up to a constant, and we were unable to find a natural way to fix this constant.

In Section 4 we compute the second ratios for τ-functions of isomonodromy transformations that reduce to dPV and $dPVI$ equations. The resulting expressions, see Theorems 4.3, 4.4, are surprisingly simple, and they should be viewed as functions on the corresponding moduli spaces of d-connections. The zeroes and poles of these second ratios show when modifications of the corresponding d-connection lead to a nontrivial vector bundle.

Section 5 is dedicated to evaluating gap probabilities for discrete biorthogonal random matrix type ensembles associated with multiple orthogonal polynomials of mixed type in the sense of [10]. This is a broad class of measures that naturally appears in a variety of domains of mathematics including enumerative combinatorics, tiling models, models of random growth, etc. In Theorem 5.3 we prove that if the difference logarithmic derivatives of all the relevant weight functions are rational, then there exists a vector bundle with a rational d-connection such that the first difference logarithmic derivatives of its τ-function (correctly defined because of certain explicit choices we make) coincide with those of the gap probabilities for the biorthogonal ensemble.

The final Section 6 provides an example: We deal with the Hahn orthogonal polynomial ensemble that comes up naturally in the statistical description of tiling of a hexagon by rhombi (see [24]) and in harmonic analysis on the infinite-dimensional unitary group (see [9]). Using the results of Sections 4 and 5, we show that the one-interval gap probability for the Hahn ensemble is expressible through a solution of the $dPVI$ equation, see Theorem 6.1. Even though a variety of results of this type are known, see [5], [4], [2], [12]-[16], [7], it is the first time that such a result involves a discrete Painlevé equation that is so high in Sakai's hierarchy. Also, this is apparently the simplest example of a model that probably cannot be handled using the isomonodromy deformations of usual connections because $dPVI$ cannot be viewed as a symmetry of a differential Painlevé equation.

Acknowledgements. The authors are very grateful to Pierre Deligne for illuminating discussions. We are particularly indebted to him for interpreting the τ-function via determinant of cohomology.

The second author (A. B.) was partially supported by the NSF grant DMS-0402047.

1. Modifications and detRΓ

1.1. Let $\mathcal{L}$ be a vector bundle on $\mathbb{P}^1$ of rank m.

Definition 1.1. A (rational) *d-connection* on $\mathcal{L}$ is a linear operator

$$\mathcal{A}(z) : \mathcal{L}_z \to \mathcal{L}_{z+1}$$

that depends on a point $z \in \mathbb{P}^1 - \{\infty\}$ in a rational way (in particular, $\mathcal{A}(z)$ is defined for all $z \in \mathbb{C}$ outside of a finite set); here $\mathcal{L}_z$ is the fiber of $\mathcal{L}$ over $z \in \mathbb{P}^1$. In other words, $\mathcal{A}$ is a rational map between the vector bundle $\mathcal{L}$ and its pullback via the automorphism $\mathbb{P}^1 \to \mathbb{P}^1$ that sends $z \mapsto z+1$.

Definition 1.2. We say that a point $z_0 \in \mathbb{P}^1$ is a *pole* of $\mathcal{A}$ if $\mathcal{A}(z)$ is not regular at $z = z_0$. We say that $z_0 \in \mathbb{P}^1$ is a *zero* of $\mathcal{A}$ if the map

$$\mathcal{A}^{-1}(z) : \mathcal{L}_{z+1} \to \mathcal{L}_z$$

is not regular at $z = z_0$. Note that $\mathcal{A}$ can have a zero and a pole at the same point.

Denote by $\operatorname{Sing}(\mathcal{A}) \subset \mathbb{C}$ the set of all zeroes and poles of $\mathcal{A}$ on $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$.

Example 1.3. Suppose that $\mathcal{A}$ has no pole at $x \in \mathbb{C}$ and that $\det(\mathcal{A})$ has a simple zero at x. Obviously x is a zero of $\mathcal{A}$. We will say that x is a *simple* zero.

Dually, suppose $\mathcal{A}$ has no zero at $x \in \mathbb{C}$ and $\det(\mathcal{A})$ has a simple pole at x. Then x is a pole of $\mathcal{A}$; we say that x is a *simple* pole.

1.2.

Definition 1.4. Suppose $\mathcal{R} : \mathcal{L} \overset{\sim}{\dashrightarrow} \mathcal{L}'$ is a rational isomorphism between two vector bundles $\mathcal{L}$ and $\mathcal{L}'$ on $\mathbb{P}^1$. We call $\mathcal{L}'$ a *modification* of $\mathcal{L}$ (of course, $\mathcal{L}$ is also a modification of $\mathcal{L}'$). If $\mathcal{R}$ is a regular map (that is, it has no poles, but it might have zeroes), we say that $\mathcal{L}'$ is an *upper* modification of $\mathcal{L}$ and $\mathcal{L}$ is a *lower* modification of $\mathcal{L}'$. For a fixed finite set $S \subset \mathbb{P}^1$, we say that $\mathcal{L}'$ is a *modification of* $\mathcal{L}$ *on* S if $\mathcal{R}(z)$ and $\mathcal{R}^{-1}(z)$ are regular outside of S.

A d-connection $\mathcal{A}$ on $\mathcal{L}$ induces a d-connection $\mathcal{A}'$ on $\mathcal{L}'$. We call $\mathcal{A}'$ a *modification* of $\mathcal{A}$.

Remark 1.5. Modifications can be viewed as an isomonodromy deformation in the sense of [6], see also [26]. Indeed, the monodromies of the difference equations associated with $\mathcal{A}$ and $\mathcal{A}'$ coincide (for the monodromies to exist, $\mathcal{A}$ and $\mathcal{A}'$ have to satisfy certain non-degeneracy conditions).

Example 1.6. Suppose $\mathcal{R} : \mathcal{L} \overset{\sim}{\dashrightarrow} \mathcal{L}'$ is regular and $\det(\mathcal{R})$ has exactly one simple zero at $x \in \mathbb{C}$. In this case, $\mathcal{L}'$ is an *elementary upper modification* (at x) of $\mathcal{L}$, and $\mathcal{L}$ is an *elementary lower modification* (at x) of $\mathcal{L}'$.

An elementary upper modification $\mathcal{R} : \mathcal{L} \to \mathcal{L}'$ at x is uniquely determined by a one-dimensional subspace $l \subset \mathcal{L}_x$ given by $l = \ker(\mathcal{R}(x) : \mathcal{L}_x \to \mathcal{L}'_x) \subset \mathcal{L}_x$. Conversely, any one-dimensional $l \subset \mathcal{L}_x$ defines an elementary upper modification at x.

Dually, elementary lower modifications of $\mathcal{L}'$ at x are in one-to-one correspondence with subspaces $l' \subset \mathcal{L}'_x$ of codimension one. (For $\mathcal{R} : \mathcal{L} \to \mathcal{L}'$, we set $l' = \operatorname{im}(\mathcal{R}(x) : \mathcal{L}_x \to \mathcal{L}'_x)$.)

1.3. Let $\mathcal{A}$ be a d-connection on $\mathcal{L}$, and suppose $x \in \operatorname{Sing}(\mathcal{A})$. Then there exists a unique modification $\mathcal{A}^{\{x\}}$ of $\mathcal{A}$ at x such that x is not a singular point of $\mathcal{A}^{\{x\}}$. Also, there exists a unique modification $\mathcal{A}_{\{x+1\}}$ of $\mathcal{A}$ at $x+1$ such that x is not a singular point of $\mathcal{A}_{\{x\}}$.

Lemma 1.7. *Suppose* $x - 1 \notin \operatorname{Sing}(\mathcal{A})$.

1. $\operatorname{Sing}(\mathcal{A}^{\{x\}}) = (\operatorname{Sing}(\mathcal{A}) \setminus \{x\}) \cup \{x-1\}$;
2. $\mathcal{A}$ *is the unique modification of* $\mathcal{A}^{\{x\}}$ *at* x *with no singularity at* $x-1$. *That is,* $\mathcal{A} = (\mathcal{A}^{\{x\}})_{\{x\}}$.

Dually, suppose $x+1 \notin \mathrm{Sing}(\mathcal{A})$.

(3) $\mathrm{Sing}(\mathcal{A}_{\{x+1\}}) = (\mathrm{Sing}(\mathcal{A}) \setminus \{x\}) \cup \{x+1\}$;

(4) $\mathcal{A}$ *is the unique modification of* $\mathcal{A}_{\{x+1\}}$ *at* x *with no singularity at* $x+1$. *That is,* $\mathcal{A} = (\mathcal{A}_{\{x+1\}})^{\{x+1\}}$.

□

Example 1.8. If $\mathcal{A}$ has a simple zero at x, then $\mathcal{A}^{\{x\}}$ is an elementary upper modification of $\mathcal{A}$ at x. This modification corresponds to the one-dimensional subspace $l = \ker(\mathcal{A}(x) : \mathcal{L}_x \to \mathcal{L}_{x+1})$ in the sense of Example 1.6.

Dually, if $\mathcal{A}$ has a simple pole at x, then $\mathcal{A}^{\{x\}}$ is the elementary lower modification of $\mathcal{A}$ at x corresponding to the codimension one subspace $\mathrm{im}(\mathcal{A}^{-1}(x) : \mathcal{L}_{x+1} \to \mathcal{L}_x)$.

1.4. Consider cohomology spaces $H^0(\mathbb{P}^1, \mathcal{L})$ and $H^1(\mathbb{P}^1, \mathcal{L})$. They have the following classical interpretation: Fix any non-empty finite set $\mathfrak{S} \subset \mathbb{P}^1$. Consider the (infinite dimensional) vector space $\Gamma(\mathcal{L}(\infty \cdot \mathfrak{S}))$ of rational sections of $\mathcal{L}$ that are allowed to have poles of any order at the points of $\mathfrak{S}$. Consider also the vector space of polar parts for rational sections of $\mathcal{L}$ at the points of $\mathfrak{S}$. It is natural to denote the space by $\Gamma(\mathcal{L}(\infty \cdot \mathfrak{S})/\mathcal{L})$.

The natural linear map

$$\Gamma(\mathcal{L}(\infty \cdot \mathfrak{S})) \to \Gamma(\mathcal{L}(\infty \cdot \mathfrak{S})/\mathcal{L})$$

sends a rational function to its polar part. The kernel of this map is identified with the space $H^0(\mathbb{P}^1, \mathcal{L})$ of global regular sections of $\mathcal{L}$. The cokernel is identified with $H^1(\mathbb{P}^1, \mathcal{L})$; this corresponds to the interpretation of classes in $H^1(\mathbb{P}^1, \mathcal{L})$ as obstructions for a Mittag-Leffler problem. Both $H^0(\mathbb{P}^1, \mathcal{L})$ and $H^1(\mathbb{P}^1, \mathcal{L})$ are finite-dimensional.

Notation. For a finite-dimensional vector space V, $\det(V)$ denotes the top exterior power of V. In particular, $\det(0) = \mathbb{C}$. If $\dim(V) = 1$, the dual of V is denoted by V^{-1}.

Definition 1.9. Define a one-dimensional vector space $\det\mathrm{R}\Gamma(\mathcal{L})$ by

$$\det\mathrm{R}\Gamma(\mathcal{L}) = \det(H^0(\mathbb{P}^1, \mathcal{L})) \otimes (\det(H^1(\mathbb{P}^1, \mathcal{L})))^{-1}.$$

For instance, suppose that $\mathcal{L} \simeq (\mathcal{O}(-1))^m$. Then $H^0(\mathbb{P}^1, \mathcal{L}) = H^1(\mathbb{P}^1, \mathcal{L}) = 0$, so $\det\mathrm{R}\Gamma(\mathcal{L}) = \mathbb{C}$ and $\det\mathrm{R}\Gamma(\mathcal{L})^{-1} = \mathbb{C}$. In particular, there is a canonical element $1 \in \det\mathrm{R}\Gamma(\mathcal{L})^{-1}$.

Definition 1.10. Suppose that a rank m vector bundle $\mathcal{L}$ has slope -1; that is, $\deg(\mathcal{L}) = -m$. We define $\tau(\mathcal{L}) \in \det\mathrm{R}\Gamma(\mathcal{L})^{-1}$ by

$$\tau(\mathcal{L}) = \begin{cases} 1 & \text{if } \mathcal{L} \simeq (\mathcal{O}(-1))^m, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.11. Suppose $\mathcal{L}$ has slope -1. Let $\mathcal{L}'$ be an arbitrary upper modification of $\mathcal{L}$. The quotient $\mathcal{L}'/\mathcal{L}$ is supported at finitely many points (the zeroes of the map $\mathcal{L} \to \mathcal{L}'$) and its space of global sections $H^0(\mathbb{P}^1, \mathcal{L}'/\mathcal{L})$ is finite-dimensional. The long exact sequence corresponding to the sequence

$$0 \to \mathcal{L} \to \mathcal{L}' \to \mathcal{L}'/\mathcal{L} \to 0$$

induces an identification between $\det\mathrm{R}\Gamma(\mathcal{L})$ and $\det(H^0(\mathbb{P}^1, \mathcal{L}')) \otimes \det(H^0(\mathbb{P}^1, \mathcal{L}'/\mathcal{L}))^{-1}$.

Consider now the natural map

$$q : H^0(\mathbb{P}^1, \mathcal{L}') \to H^0(\mathbb{P}^1, \mathcal{L}'/\mathcal{L}).$$

Its determinant

$$\det(q) \in \det(H^0(\mathbb{P}^1, \mathcal{L}'/\mathcal{L})) \otimes \det(H^0(\mathbb{P}^1, \mathcal{L}'))^{-1} = \det\mathrm{R}\Gamma(\mathcal{L})^{-1}$$

equals $\tau(\mathcal{L})$. In particular, q is an isomorphism if and only if $\mathcal{L} \simeq (\mathcal{O}(-1))^m$.

Example 1.12. Let $\mathcal{L}_2$ be a modification of $\mathcal{L}_1$. Suppose that $\mathcal{L}_2$ has slope -1 and $\mathcal{L}_1 \simeq (\mathcal{O}(-1))^m$. One can choose an upper modification $\mathcal{L}'$ of $\mathcal{L}_1$ that is also an upper modification of $\mathcal{L}_2$. Consider the ratio

$$\frac{\tau(\mathcal{L}_2)}{\tau(\mathcal{L}_1)} \in \det\mathrm{R}\Gamma(\mathcal{L}_1) \otimes \det\mathrm{R}\Gamma(\mathcal{L}_2)^{-1}.$$

By the previous example, we can identify

$$\det\mathrm{R}\Gamma(\mathcal{L}_1) \otimes \det\mathrm{R}\Gamma(\mathcal{L}_2)^{-1} \text{ and } \det(H^0(\mathbb{P}^1, \mathcal{L}'/\mathcal{L}_2)) \otimes \det(H^0(\mathbb{P}^1, \mathcal{L}'/\mathcal{L}_1))^{-1}.$$

Under this identification, the ratio corresponds to the determinant of the composition

$$H^0(\mathbb{P}^1, \mathcal{L}'/\mathcal{L}_1) \xrightarrow{\sim} H^0(\mathbb{P}^1, \mathcal{L}') \xrightarrow{\sim} H^0(\mathbb{P}^1, \mathcal{L}'/\mathcal{L}_2). \tag{1.1}$$

2. Ratios of the τ-function

In this section, we study τ-functions of vector bundles with d-connections. First, we consider d-connections with arbitrary singularities, and then look at three special cases.

2.1. Let $\mathcal{L}$ be a rank m vector bundle and $\mathcal{A}$ be a d-connection on $\mathcal{L}$. Suppose that $\mathcal{A}$ has singularities at n distinct points $a_1, \dots, a_n$ and no singularities at $a_i + k$ for $k \in \mathbb{Z} - \{0\}$, $i = 1, \dots, n$. We impose no restrictions on the behavior of $\mathcal{A}$ elsewhere.

Consider the lattice

$$\Lambda := (a_1, \dots, a_n) + \mathbb{Z}^n \subset \mathbb{C}^n.$$

Fix $u = (u_1, \dots, u_n) \in \Lambda$. By virtue of Lemma 1.7, there exists a unique modification $\mathcal{L}_u$ of $\mathcal{L}$ at a subset of Λ such that the d-connection $\mathcal{A}_u$ on $\mathcal{L}_u$ satisfies

$$(\mathrm{Sing}(\mathcal{A}_u) = \mathrm{Sing}(\mathcal{A}) \setminus \{a_1, \dots, a_n\}) \cup \{u_1, \dots, u_n\}.$$

In other words, the singularities at a_i's are shifted to u_i's.

For $u \in \Lambda$,

$$\deg(\mathcal{L}_u) = \deg(\mathcal{L}) - \sum \kappa_i(u_i - a_i),$$

where κ_i is the order of zero of $\det(\mathcal{A})$ at a_i (κ_i can be negative). Consider the sublattice

$$\Lambda_{-m} := \{u \in \Lambda : \deg(\mathcal{L}_u) = -m\} \subset \Lambda.$$

Set $T_u := \det\mathrm{R}\Gamma(\mathcal{L}_u)^{-1}$. This is a one-dimensional vector space depending on $u \in \Lambda$; if $u \in \Lambda_{-m}$, we have a natural element $\tau(\mathcal{L}_u) \in T_u$.

Note that according to our definition, τ is not a function on Λ_{-m}, because its value belongs to a one-dimensional vector space that has no natural basis. Nevertheless, it turns out that the 'second logarithmic derivative' of τ makes sense as a function on Λ_{-m}. Let us make the statement precise:

Proposition 2.1. *For $i = 1, \dots, n$, let $e_i \in \mathbb{Z}^n$ be the ith standard basis vector. Then the 'first derivative'*

$$S_u^{(i)} := T_u \otimes (T_{u-e_i})^{-1}$$

does not depend on $u \in \Lambda$. That is, there exists a canonical isomorphism $S_u^{(i)} \widetilde{\to} S_v^{(i)}$ for any $u, v \in \Lambda$.

Proof. Fix $u = (u_1, \dots, u_n) \in \Lambda$ and $i = 1, \dots, n$. The bundle $\mathcal{L}_{u-e_i}$ is a modification of $\mathcal{L}_u$ at u_i; let us write it as a combination of an upper modification $\mathcal{L}_u \to \mathcal{L}'_u$ and a lower modification $\mathcal{L}'_u \leftarrow \mathcal{L}_{u-e_i}$ at u_i. As in Example 1.12, we have an isomorphism
(2.1)

$$S_u^{(i)} = \det\mathrm{R}\Gamma(\mathcal{L}_{u-e_i}) \otimes \det\mathrm{R}\Gamma(\mathcal{L})^{-1} = \det(H^0(\mathbb{P}^1, \mathcal{L}'_u/\mathcal{L}_u)) \otimes \det(H^0(\mathbb{P}^1, \mathcal{L}'_u/\mathcal{L}_{u-e_i}))^{-1}.$$

From this description of $S_u^{(i)}$, one immediately gets an identification $S_u^{(i)} \widetilde{\to} S_v^{(i)}$ provided u and v have equal ith component. Indeed, $\mathcal{L}_u$ and $\mathcal{L}_v$ are naturally identified in the neighborhood of $u_i = v_i$; denote the identification by ϕ. There is a unique upper modification $\mathcal{L}'_v$ of $\mathcal{L}_v$ at u_i such that ϕ identifies $\mathcal{L}'_u$ and $\mathcal{L}'_v$ near u_i. Then $\mathcal{L}_{v-e_i}$ is a lower modification of $\mathcal{L}'_v$, and ϕ is also an isomorphism between $\mathcal{L}_{u-e_i}$ and $\mathcal{L}_{v-e_i}$ near v_i. Therefore, ϕ induces an isomorphism between the right-hand side of (2.1) and the corresponding formula for $S_v^{(i)}$.

It remains to construct an isomorphism $S_u^{(i)} \widetilde{\to} S_{u-e_i}^{(i)}$. For z close to $u_i - 1$, the map $\mathcal{A}_u(z)$ is an isomorphism between $\mathcal{L}_{u-e_i}$ near $u_i - 1$ and $\mathcal{L}_u$ near u_i. Indeed, $\mathcal{L}_{u-e_i}$ coincides with $\mathcal{L}_u$ near $u_i - 1$, while $\mathcal{A}_u$ has no singularity at $z = u_i - 1$. There is a unique upper modification $\mathcal{L}'_{u-e_i}$ of $\mathcal{L}_{u-e_i}$ at $u_i - 1$ such that $\mathcal{A}_u$ identifies $\mathcal{L}'_{u-e_i}$ near $u_i - 1$ with $\mathcal{L}'_u$ near u_i. Then $\mathcal{L}_{u-2e_i}$ is a lower modification of $\mathcal{L}'_{u-e_i}$, and $\mathcal{A}_u$ is also an isomorphism between $\mathcal{L}_{u-2e_i}$ near $u_i - 1$ and $\mathcal{L}_{u-e_i}$ near u_i. Therefore, $\mathcal{A}_u$ induces an isomorphism between the right-hand side of (2.1) and the corresponding formula for $S_{u-e_i}^{(i)}$.

It is easy to see that the constructed isomorphisms $S_u^{(i)} \widetilde{\to} S_v^{(i)}$ do not depend on the choice of the upper modification $\mathcal{L}_u \to \mathcal{L}'_u$. □

Since $S_u^{(i)}$ does not depend on $u \in \Lambda$, we suppress the index u from now on.

Let us now define the ratios of τ. Set

$$\mathbb{Z}_0^n = \{s = (s_1, \dots, s_n) : \sum_{i=1}^n \kappa_i s_i = 0\}.$$

For any $u \in \Lambda_{-m}$, $s \in \mathbb{Z}_0^n$, we have $u + s \in \Lambda_{-m}$. Consider the ratio

$$D_s\tau(u) := \frac{\tau(u+s)}{\tau(u)} \in T_{u+s} \otimes T_u^{-1} = \bigotimes_i (S^{(i)})^{\otimes s_i}.$$

Now fixing $t \in \mathbb{Z}_0^n$, we see that the second ratio

$$D_{s,t}\tau(u) := \frac{D_s\tau(u+t)}{D_s(u)}$$

makes sense as a number.

2.2. **Sign issues.** Let us choose bases in vector spaces $S^{(i)}$ for $i = 1, \dots, m$. Then for any $s \in \mathbb{Z}_0^n$, $u \in \Lambda_{-m}$, the ratio $D_s\tau(u)$ becomes a number. It is tempting to say that these numbers are partial derivatives of a function

$$\tilde{\tau} : \Lambda_{-m} \to \mathbb{C}.$$

To construct such $\tilde{\tau}$, we need to choose bases in vector spaces T_u (for all $u \in \Lambda_{-m}$) consistent with the bases in $S^{(i)}$ in the sense of Proposition 2.1.

Generally speaking, this is impossible. More precisely, one can choose bases in T_u that are consistent up to sign. Equivalently, $\tilde{\tau}$ can be defined as a function on a two-fold cover of Λ_{-m}. Let us explain the sign in more details.

The basic reason for the sign is that for two finite-dimensional vector spaces V, W, the isomorphism $\det(V \oplus W) \simeq \det(V) \otimes \det(W)$ agrees with permutation up to sign only. Indeed, the composition

$$\det(V)\otimes\det(W) \simeq \det(V\oplus W) \simeq \det(W\oplus V) \simeq \det(W)\otimes\det(V) \simeq \det(V)\otimes\det(W)$$

equals $(-1)^{\dim(V)\dim(W)}$. As a result, when we use Proposition 2.1 to identify

$$T_u \otimes (T_{u-e_i-e_j})^{-1} = (T_u \otimes (T_{u-e_i})^{-1}) \otimes (T_{u-e_i} \otimes (T_{u-e_i-e_j})^{-1}) = S^{(i)} \otimes S^{(j)},$$

the identification is multiplied by $(-1)^{\kappa_i\kappa_j}$ when i and j are permuted (assuming $i \neq j$).

Denote by $\widetilde{\mathbb{Z}}^n$ the group generated by $\tilde{e}_i$ $(i = 1, \dots, n)$ and ϵ subject to relations

$$2 \cdot \epsilon = 0, \quad \epsilon \dotplus \tilde{e}_i = \tilde{e}_i \dotplus \epsilon, \quad \tilde{e}_i \dotplus \tilde{e}_j = (\kappa_i\kappa_j)\epsilon \dotplus \tilde{e}_j \dotplus \tilde{e}_i.$$

Recall that κ_i is the order of zero of $\det(\mathcal{A})$ at a_i. Consider the homomorphism $\pi : \widetilde{\mathbb{Z}}^n \to \mathbb{Z}^n$ that sends $\tilde{e}_i$ to e_i and ϵ to 0. Using π, we can view $\widetilde{\mathbb{Z}}^n$ as a central extension of $\mathbb{Z}^n$ by $\{0, \epsilon\}$. The group $\mathbb{Z}^n$ acts on Λ; the action lifts to a natural action of $\widetilde{\mathbb{Z}}^n$ on $\{T_u\}_{u \in \Lambda}$. That is, for $u, v \in \Lambda$, an isomorphism $T_u \xrightarrow{\sim} T_v$ is determined by an element of $\pi^{-1}(v - u) \subset \widetilde{\mathbb{Z}}^n$.

Set $\widetilde{\mathbb{Z}}_0^n := \pi^{-1}(\mathbb{Z}_0^n)$. Fix a basis in T_u for single $u \in \Lambda_{-m}$. Acting by $\tilde{s} \in \widetilde{\mathbb{Z}}_0^n$, we obtain a basis in $T_{u+\pi(\tilde{s})}$. Therefore, $\tilde{\tau}$ is well defined as a function on the set

$$\widetilde{\Lambda}_{-m} := \{u \dotplus \tilde{s} : \tilde{s} \in \widetilde{\mathbb{Z}}_0^n\}.$$

Remark. The situation simplifies if all κ_i's are even. In this case, the central extension $\widetilde{\mathbb{Z}}^n$ splits, and $\tilde{\tau}$ makes sense as a function on Λ_{-m}. An example of such situation is considered in Section 2.5.

2.3. **d-connections with simple zeroes.** Suppose that $\mathcal{A}$ has a simple zero at $z = a_i$ for $i = 1, \dots, n$. Let us make the construction of Section 2.1 more explicit in this case.

For $u \in \Lambda$, $i = 1, \dots, n$, $\mathcal{L}_{u-e_i}$ is an elementary upper modification of $\mathcal{L}_u$ at u_i. The modification is given by a dimension one subspace $l = l_{u,i}$ in the fiber of $\mathcal{L}_u$ at u_i (see Example 1.6). By Example 1.8, l is the kernel of the operator

$$\mathcal{A}_u(u_i) : (\mathcal{L}_u)_{u_i} \to (\mathcal{L}_u)_{u_i+1}.$$

As in the proof of Proposition 2.1, we have an isomorphism

$$l = \det\mathrm{R}\Gamma(\mathcal{L}_{u-e_i}) \otimes \det\mathrm{R}\Gamma(\mathcal{L}_u)^{-1} = S_u^{(i)} \tag{2.2}$$

(this corresponds to taking $\mathcal{L}'_u = \mathcal{L}_{u-e_i}$).

Set $s = e_i - e_j \in \mathbb{Z}_n^0$, and let us give explicit formulas for the ratio

$$D_s\tau(u) = \frac{\tau(u + e_i - e_j)}{\tau(u)} \in S^{(i)} \otimes (S^{(j)})^{-1}.$$

Fix $u \in \Lambda_{-m}$, and let $l = l_{u,j} \subset (\mathcal{L}_u)_{u_j}$ be the dimension one subspace corresponding to the upper modification $\mathcal{L}_{u-e_j}$ of $\mathcal{L}$. By (2.2), $S^{(j)} = l$.

Similarly, let us consider $\mathcal{L}_{u+e_i}$ as a lower modification of $\mathcal{L}_u$ (instead of viewing $\mathcal{L}_u$ as an upper modification of $\mathcal{L}_{u+e_i}$). Let $l' = l'_{u,i} \subset (\mathcal{L}_u)_{u_i+1}$ be the codimension one subspace corresponding to this modification. By the proof of Proposition 2.1, we get an identification $S^{(i)} = (\mathcal{L}_u)_{u_i+1}/l'$.

The identifications $S^{(i)} = l_{u+e_i,i}$ and $S^{(i)} = (\mathcal{L}_u)_{u_i+1}/l'$ are related as follows. Consider multiplication by $z - (u_i + 1)$; it gives a morphism $\mathcal{L}_u \to \mathcal{L}_{u+e_i}$ (with a pole at infinity). Taking the value of this morphism at $z = u_i + 1$, we obtain a linear operator $(\mathcal{L}_u)_{u_i+1} \to (\mathcal{L}_{u+e_i})_{u_i+1}$. It is easy to see that the operator factors into a composition

$$(\mathcal{L}_u)_{u_i+1} \to (\mathcal{L}_u)_{u_i+1}/l' \xrightarrow{\sim} l_{u+e_i,i} \hookrightarrow (\mathcal{L}_{u+e_i})_u. \tag{2.3}$$

This provides an isomorphism between $l_{u+e_i,i}$ and $(\mathcal{L}_u)_{u_i+1}/l'$.

The isomorphism (2.3) becomes more natural if we use the d-connection to identify $l_{u,i}$ with $l_{u+e_i,i}$ (as in Proposition 2.1). Specifically, we obtain an isomorphism

$$l_{u,i} \xrightarrow{\sim} (\mathcal{L}_u)_{u_i+1}/l' \quad : \quad w \mapsto \frac{d\mathcal{A}_u(u_i)}{dz} w. \tag{2.4}$$

Suppose now that $\mathcal{L}_u \simeq (\mathcal{O}(-1))^m$. Fixing an isomorphism $\iota : \mathcal{L}_u \simeq (\mathcal{O}(-1))^m$, we can identify the fiber $(\mathcal{L}_u)_z$ with $(\mathcal{O}(-1))^m_z = \mathbb{C}^m$ for any point $z \in \mathbb{C}$. In particular, the fibers $(\mathcal{L}_u)_{u_j}$ and $(\mathcal{L}_u)_{u_i+1}$ are identified. Actually, this identification does not depend on the choice of isomorphism ι.

Proposition 2.2. *The composition*

$$S^{(j)} = l \hookrightarrow (\mathcal{L}_u)_{u_j} \xrightarrow{\sim} (\mathcal{L}_u)_{u_i+1} \to (\mathcal{L}_u)_{u_i+1}/l' = S^{(i)}$$

is equal to multiplication by

$$(u_i + 1 - u_j)\frac{\tau(u + e_i - e_j)}{\tau(u)} = (u_i + 1 - u_j) D_s\tau(u)$$

Proof. By Example 1.12, the first derivative $D_s\tau(u)$ is equal to the determinant of the composition

$$H^0(\mathbb{P}^1, \mathcal{L}_{u-e_j}/\mathcal{L}_u) \xrightarrow{\sim} H^0(\mathbb{P}^1, \mathcal{L}_{u-e_j}) \xrightarrow{\sim} H^0(\mathbb{P}^1, \mathcal{L}_{u-e_j}/\mathcal{L}_{u+e_i-e_j}). \tag{2.5}$$

We can identify $H^0(\mathbb{P}^1, \mathcal{L}_{u-e_j}/\mathcal{L}_u)$ with l and $H^0(\mathbb{P}^1, \mathcal{L}_{u-e_j}/\mathcal{L}_{u+e_i-e_j})$ with $(\mathcal{L}_{u-e_j})_{u_i+1}/l'_{u-e_j,i} = (\mathcal{L}_u)_{u_i+1}/l'$, where the last equality follows from the fact that $\mathcal{L}_u$ and $\mathcal{L}_{u-e_j}$ coincide near $u_i + 1$. Finally, global sections of $\mathcal{L}_{u-e_j}$ are of the form

$$\frac{\lambda}{z - u_j}, \qquad \lambda \in l \subset (\mathcal{L}_u)_{u_j} = \mathbb{C}^m$$

(we are using the identification $\iota : \mathcal{L}_u \simeq (\mathcal{O}(-1))^m$ here). ◻

Let us rewrite Proposition 2.2 using explicit coordinates. Fix an isomorphism $\iota : \mathcal{L}_u \simeq (\mathcal{O}(-1))^m$. The d-connection $\mathcal{A}_u$ is then given by its matrix $A_u(z)$. By assumption, $A_u(z)$ is regular at all points $a_i+\mathbb{Z}$, $i = 1, \dots, n$. Also, $\det(A_u(z))$ has simple zeroes at u_i and no zeroes at $u_i + (\mathbb{Z} - \{0\})$, $i = 1, \dots, n$.

Using ι, we identify $l_{u,j} \subset (\mathcal{L}_u)_{u_j}$ with the kernel of $A_u(u_j)$. Similarly, $l'_{u,i} \subset (\mathcal{L}_u)_{u_i+1}$ is identified with the image of $A_u(u_i)$. Instead of working with the codimension one subspace $l'_{u,i} \subset \mathbb{C}^n$, we can consider its orthogonal complement, which is a one-dimensional subspace $(l'_{u,i})^\perp \subset (\mathbb{C}^n)^* = \mathbb{C}^n$.

Pick bases $w = w_{u,j} \in l_{u,j}$ and $w' = w'_{u,i} \in (l'_{u,i})^\perp$. We have the following formula for $A_{u+e_i-e_j}(z)$:
(2.6)

$$A_{u+e_i-e_j}(z) = R_{u,i,j}(z+1)A_u(z)R_{u,i,j}(z)^{-1},$$

$$R_{u,i,j}(z) = I + \frac{R_0}{z-u_i-1}, \quad R_{u,i,j}(z)^{-1} = I - \frac{R_0}{z-u_j}, \quad \det R_{u,i,j}(z) = \frac{z-u_j}{z-u_i-1},$$

$$R_0 = \frac{u_i-u_j+1}{\langle w, w'\rangle}\, w \cdot (w')^t.$$

Observe that $A_{u+e_i-e_j}$ and $R_{u,i,j}$ are independent of the choice of w and w'.

In Proposition 2.1, we show that for fixed k, the vector space $l_{u,k}$ does not depend on $u \in \Lambda_{-m}$ in the sense that the corresponding spaces are related by natural isomorphisms. The isomorphisms $l_{u,k} \xrightarrow{\sim} l_{u+e_i-e_j,k}$ are given by the following formulas:

$$w \mapsto \begin{cases} R_{u,i,j}(u_k)\cdot w, & k \neq i, j; \\ R_{u,i,j}(u_j-1)A_u(u_j-1)^{-1}\cdot w, & k = j; \\ R_{u,i,j}(u_i+1)^{-1}A_{u+e_i-e_j}(u_i)\cdot w, & k = i. \end{cases} \tag{2.7}$$

It is not hard to check explicitly that the isomorphisms are consistent. Therefore, a choice of a basis $w \in l_{u,k}$ for one $u \in \Lambda_{-m}$ determines bases $w_{v,k} \in l_{v,k}$ for all $v \in \Lambda_{-m}$. Let us fix these bases.

Dually, the vector spaces $(l'_{u,k})^\perp$ are identified for all $u \in \Lambda_{-m}$. The identifications are given by formulas similar to (2.7), which can be obtained using (2.4). Denote by $w'_{u,k} \in l'_{u,k}$ the basis dual to $w_{u,k}$.

We can now rewrite Proposition 2.2 as the following formula:

$$\frac{\tilde\tau(u \dotplus \tilde e_i \dot- \tilde e_j)}{\tilde\tau(u)} = \frac{\langle w_{u,j}, w'_{u,i}\rangle}{u_i+1-u_j}.$$

Clearly, the second derivatives of $\tilde\tau$ are independent of all choices.

These formulas can be used in a more 'classical' definition of the τ function as a solution to a system of difference equations. From this point of view, the existence of a solution is not obvious; this leads to the following statement.

Corollary 2.3. *Let $A(z)$ be a square matrix with rational entries that is regular at points $a_i + \mathbb{Z}$, $i = 1, \dots, n$. Assume that $\det(A(z))$ has simple zeroes at a_i and no zeroes at $a_i + (\mathbb{Z} - \{0\})$, $i = 1, \dots, n$.*

For any $u = (u_1, \dots, u_n) \in (a_1, \dots, a_n) + \mathbb{Z}^n$ with $\sum u_i = \sum a_i$, we define the isomonodromy deformation $A_u(z)$ recursively using formulas (2.6).

1. *$A_u(z)$ is well defined, provided $A(z)$ is generic in the sense that $\langle w, w'\rangle$ does not vanish in (2.6). In particular, $A_u(z)$ does not depend on a representation of $u - a$ as a linear combination of generators $e_i - e_j$.*

(2) *Choose bases* $w_{u,k}$, $w'_{u,k}$ *as above. There exists a function* $\tilde{\tau}(a \dot{+} \tilde{s})$, *where* $\tilde{s} \in \widetilde{\mathbb{Z}}_0^n$, *such that*

$$\frac{\tilde{\tau}(\tilde{u} \dot{+} \tilde{e}_i \dot{-} \tilde{e}_j)}{\tilde{\tau}(\tilde{u})} = \frac{\langle w_{u,j}, w'_{u,i}\rangle}{u_i + 1 - u_j}$$

for every $\tilde{u} = a \dot{+} \tilde{s}$, $\tilde{u} \in \widetilde{\mathbb{Z}}_0^n$. *Here* $u = \pi(\tilde{u}) = a + \pi(\tilde{s})$.

Proof. (1) follows from Lemma 1.7; (2) follows from Proposition 2.2. □

Remark. Corollary 2.3(1) is a version of Theorem 2.1 of [6].

2.4. **d-connections with simple zeroes and poles.** Now suppose the d-connection has simple zeroes and simple poles. Let us consider isomonodromy deformations corresponding to a simultaneous shift of a simple pole and a simple zero. This kind of equations is important because it has one of the simplest continuous limits (Section 3). Let us give an analog of Corollary 2.3; the proof is completely similar. We omit the coordinate-free formulation (an analog of Proposition 2.2).

Suppose $A(z)$ has simple zeroes at n_a distinct points $a_1, \dots, a_{n_a}$, simple poles at n_b distinct points $b_1, \dots, b_{n_b}$, and no singularities at $a_i + (\mathbb{Z} - \{0\})$, $b_j + (\mathbb{Z} - \{0\})$. Suppose

$$(u; v) = (u_1, \dots, u_{n_a}; v_1, \dots, v_{n_b}) \in (a_1, \dots, a_{n_a}; b_1, \dots, b_{n_b}) + \mathbb{Z}^{n_a+n_b}$$

satisfies

$$\sum (u_i - a_i) = \sum (v_j - b_j).$$

Assuming $A(z)$ is generic, there is a unique matrix $R(z)$ with rational coefficients that satisfies the following conditions:

(1) All singularities of $R(z)$ and $R^{-1}(z)$ belong to the progressions $a_i + \mathbb{Z}$, $b_j + \mathbb{Z}$;
(2) $R(\infty) = I$;
(3) $A_{u;v}(z) = R(z+1)A(z)R(z)^{-1}$ has simple zeroes at $u_1, \dots, u_{n_a}$, simple poles at $v_1, \dots, v_{n_b}$, and no singularities at $u_i + (\mathbb{Z} - \{0\})$, $v_j + (\mathbb{Z} - \{0\})$.

For instance, if $u = (a_1 - 1, a_2, \dots, a_{n_a})$ and $v = (b_1 - 1, b_2, \dots, b_{n_b})$, $R(z)$ is given by the following formulas:

$$\begin{gathered} R(z) = I + \frac{R_0}{z - b_1}, \quad R(z)^{-1} = I - \frac{R_0}{z - a_1}, \quad \det R(z) = \frac{z - a_1}{z - b_1}, \\ R_0 = \frac{b_1 - a_1}{\langle w, w'\rangle} w \cdot (w')^t. \end{gathered} \tag{2.8}$$

Here w is a basis in the kernel of $A(a_1)$, and w' is a basis in the image of $\lim_{z\to b_1}(z - b_1)A^t(z)$. Similar formulas can be found for other 'elementary shifts'. One can then use these formulas to compute $A_{u;v}(z)$ recursively.

Similarly to (2.7), a choice of w and w' for all singular points of $A(z)$ determines bases in the corresponding spaces for all deformations $A_{u;v}(z)$. This allows us to consider τ as a function of $(u; v)$.

Similarly to Section 2.2, the function τ is defined only up to a sign (that is, it is a function on the two-fold cover of the set of $(u; v)$). One way to avoid this complication is to assume that $n_a = n_b = n$ and that we always move ith zero and

ith pole simultaneously. Let us make this assumption, so that $u_i - a_i = v_i - b_i$ for all i. Then the function τ satisfies the following equation:

$$\frac{\tau(u_1 - 1, u_2, \dots, u_n; v_1 - 1, v_2, \dots, v_n)}{\tau(u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_n)} = \frac{\langle w, w' \rangle}{u_1 - v_1}.$$

Here w is the basis in the kernel of $A_{u;v}(u_1)$, and w' is the basis in the image of $\lim_{z \to v_1}(z - v_1)A^t_{u;v}(z)$. As above, the second derivatives are independent of the choice of w and w'.

2.5. Let us now look at another type of singularity structure that becomes useful in Section 5. Suppose $A(z)$ has singularities at n points $a_1, \dots, a_n$, and no singularities at $a_i + (\mathbb{Z} - \{0\})$. The singularities at a_i are of the following kind:

- Matrix elements of $A(z)$ have at most a first-order pole at a_i;
- $\operatorname{res}_{a_i}(A(z))$ is a matrix of rank 1;
- $\det(A(z))$ is regular nonzero at a_i.

This can be viewed as a degeneration of the situation considered in the previous section, when zeroes and poles coalesce.

If $A(z)$ is generic, for $(u_1, \dots, u_n) \in (a_1, \dots, a_n) + \mathbb{Z}^n$, there exists a unique rational matrix $R(z) = R_u(z)$ with the following properties:

(1) All singularities of $R(z)$ and $R^{-1}(z)$ belong to the progressions $a_i + \mathbb{Z}$;
(2) $R(\infty) = I$;
(3) $A_u(z) = R(z + 1)A(z)R(z)^{-1}$ has the same singularity structure as $A(z)$ with singularities at u_i.

Choose a basis $w = w_u$ in the image of $\operatorname{res}_{u_1}(A_u^{-1}(z))$, and a functional $w' + w''(z - u_1) = w'_u + w''_u(z - u_1)$ such that

$$\langle w, w' \rangle = 0 \tag{2.9}$$

$$w' \neq 0 \tag{2.10}$$

$$A_u^{-t}(z)(w' + w''(z - u_1)) \text{ vanishes at } z = u_1. \tag{2.11}$$

Note that (2.9) implies that (2.11) is regular at $z = u_1$. Equivalently, (2.9)–(2.11) mean that in a neighborhood of $z = u_1$, we can write

$$A(z) = H(z)\left(I + \frac{1}{z - u_1} \cdot \frac{w \cdot (w')^t}{\langle w, w'' \rangle}\right)$$

for a holomorphic invertible matrix $H(z)$. The pair $(w', w'' \mod w^\perp)$ is defined up to a scalar.

Remark. Geometrically, the choices can be explained in terms of Section 2.1. Suppose the d-connection $\mathcal{A}_u$ on a vector bundle $\mathcal{L}_u$ has at $z = u_1$ a singularity of the kind we consider. There exists a unique elementary upper modification $\mathcal{L}'_u$ of $\mathcal{L}_u$ at u_1 such that $\mathcal{L}_{u-e_1}$ is an elementary lower modification of $\mathcal{L}'_u$ at u_1. The vector w is a basis in the dimension one space $l \subset (\mathcal{L}_u)_{u_1}$ corresponding to the modification $\mathcal{L}_u \to \mathcal{L}'_u$, while $w' + w''(z - u_1)$ should be thought of as a functional on the fiber $(\mathcal{L}'_u)_{u_1}$ whose kernel is the codimension one space $l' \subset (\mathcal{L}'_u)_{u_1}$ corresponding to the modification $\mathcal{L}'_u \leftarrow \mathcal{L}_{u-e_1}$.

For $u = (a_1 - 1, a_2, \dots, a_n)$, we can write $R(z)$ in terms of w, w', w'':

$$R(z) = I + \frac{R_0}{z - a_1}\,, \quad R(z)^{-1} = I - \frac{R_0}{z - a_1}\,, \quad \det R(z) = 1\,,$$
$$R_0 = \frac{w \cdot (w')^t}{\langle w, w'' \rangle}\,.$$

The choice of $(w, w', w'' \mod w^{\perp})$ for $A(z)$ determines corresponding choices $(w_u, w'_u, w''_u \mod w_u^{\perp})$ for all deformations $A_u(z)$. Explicitly, if $u_1 = a_1$, we have

$$w_u = R_u(a_1) w, \qquad w'_u = R_u^{-t}(a_1) w',$$
$$w''_u = R_u^{-t}(a_1) w'' + \left.\frac{dR_u^{-t}(z)}{dz}\right|_{z=a_1} w'.$$

Here $R_u(z)$ is the gauge matrix: $A_u(z) = R_u(z+1)A(z)R_u(z)^{-1}$. On the other hand, for $u = (a_1 - 1, a_2, \dots, a_n)$, we have

$$w_u = R_u(a_1-1)A^{-1}(a_1-1)w, \qquad w'_u = R_u^{-t}(a_1-1)A^t(a_1-1)w',$$
$$w''_u = R_u^{-t}(a_1-1)A^t(a_1-1)w'' + \left.\frac{d(R_u^{-t}(z)A^t(z))}{dz}\right|_{z=a_1-1} w'.$$

Similarly, we choose triples $(w, w', w'' \mod w^{\perp})$ at other singularities $a_2, \dots, a_n$ of $A(z)$, and obtain corresponding triples for all deformations $A_u(z)$. After these choices, we can view τ as a function of u. Note that τ is defined canonically (not just up to sign), because zeroes and poles move in pairs (in terms of Section 2.1, $\kappa_i = 0$). The equation for τ then becomes

$$\frac{\tau(u_1 - 1, u_2, \dots, u_n)}{\tau(u_1, \dots, u_n)} = \langle w_u, w''_u \rangle.$$

(This is the value of functional $w'_u + w''_u(z - u_1)$ on section $w/(z - u_1)$ of the elementary upper modification of the original bundle.)

2.6. **Hirota identities.** τ-functions satisfy various determinantal identities of Hirota type. Let us show how they arise from isomonodromy transformations. To be concrete, we restrict ourselves to the case when $\mathcal{A}$ has simple zeroes. Another case, when singularities are of type considered in Section 2.5, appears in Section 5, see Remark 5.4.

Proposition 2.4. *In the settings of Section 2.3, assume that* $\mathcal{L}_u \simeq (\mathcal{O}(-1))^m$. *Then*

$$\frac{\tau\left(u + \sum\limits_{i \in I} e_i - \sum\limits_{j \in J} e_j\right)}{\tau(u)} = \det\left[\frac{\tau(u + e_i - e_j)}{\tau(u)}\right]_{i \in I, j \in J}. \tag{2.12}$$

Here $I, J \subset \{1, \dots, n\}$ *are non-intersecting subsets of the same cardinality.*

Remark. As explained in Section 2.2, $\tau(u+s)/\tau(u)$ is defined up to a sign, corresponding to the lift of $s \in \mathbb{Z}^m$ to the two-fold cover $\widetilde{\mathbb{Z}}^m$. In Proposition 2.4, the lift is chosen as follows: for orderings $i_1, \dots, i_k$ of I and $j_1, \dots, j_k$ of J, we take

$$\tilde{e}_{i_1} \dot{-} \tilde{e}_{j_1} \dot{+} \dots \dot{+} \tilde{e}_{i_k} \dot{-} \tilde{e}_{j_k}$$

as the lift of $\sum e_i - \sum e_j$ in the left-hand side, and $\tilde{e}_i \dot{-} \tilde{e}_j$ as the lift of $e_i - e_j$ in the right-hand side. The orderings of I and J also fix the order of rows and columns in the determinant.

Proof. Let us choose bases $w_{u,k}$, $w'_{u,k}$ as in Section 2.3. By Example 1.12, the left-hand side of (2.12) equals the determinant of the transition matrix in $H^0(\mathbb{P}^1, \mathcal{L}_{u-\sum e_j})$ from the basis consisting of meromorphic sections of $\mathcal{L}_u$ with a single pole at u_j and residue $w_{u,j}$ to the dual basis of the functionals that send a section s to $w'_{u,i}(s(u_i+1))$. (These bases come from $H^0(\mathbb{P}^1, \mathcal{L}_{u-\sum e_j}/\mathcal{L})$ and $H^0(\mathbb{P}^1, \mathcal{L}_{u-\sum e_j}/\mathcal{L}_{u+\sum e_i-\sum e_j})$, respectively.)

Explicitly, we can choose a trivialization $\mathcal{L}_u \simeq (\mathcal{O}(-1))^m$, and then

$$\frac{\tau\left(u+\sum\limits_{i\in I} e_i - \sum\limits_{j\in J} e_j\right)}{\tau(u)} = \det\left[\frac{\langle w_{u,j}, w'_{u,i}\rangle}{u_i+1-u_j}\right]_{i\in I, j\in J}.$$

The statement follows. □

Remark. Suppose $I=\{i_1,i_2\}$, $J=\{j_1,j_2\}$. Then (2.12) takes the form

$$\begin{aligned}&\tilde{\tau}(u \dot{+} e_{i_1} \dot{-} e_{j_1} \dot{+} e_{i_2} \dot{-} e_{j_2})\,\tilde{\tau}(u)\\ &\qquad = \tilde{\tau}(u \dot{+} e_{i_1} \dot{-} e_{j_1})\,\tilde{\tau}(u \dot{+} e_{i_2} \dot{-} e_{j_2}) - \tilde{\tau}(u \dot{+} e_{i_1} \dot{-} e_{j_2})\,\tilde{\tau}(u \dot{+} e_{i_2} \dot{-} e_{j_1}).\end{aligned}$$

This is often called Hirota's difference bilinear equation.

3. Continuous limit

3.1. Let us recall some properties of isomonodromy deformation of connections on $\mathbb{P}^1$ in the simplest case of regular singularities.

Consider the system of linear ordinary differential equations

$$\frac{dY(\zeta)}{d\zeta} = B(\zeta)Y(\zeta), \quad B(\zeta) = \sum_{i=1}^{n} \frac{B_i}{\zeta - y_i}, \tag{3.1}$$

where B_i's are constant $m\times m$ matrices. Clearly, (3.1) has regular singularities (simple poles) at $\zeta = y_1, \dots, y_n, \infty$ and no other poles. Geometrically, we can view (3.1) as a connection on the trivial vector bundle on $\mathbb{P}^1$.

The isomonodromy deformation of (3.1) is controlled by a system of differential equations on B_i's (viewed as functions of y_j's) called the *Schlesinger system*:

$$\frac{\partial B_i}{\partial y_j} = \frac{[B_i, B_j]}{y_i - y_j}, \quad \frac{\partial B_i}{\partial y_i} = -\sum_{j\neq i} \frac{[B_i, B_j]}{y_i - y_j}. \tag{3.2}$$

Instead of working with B_i, let us consider $B(\zeta)$ given by (3.1). Then (3.2) can be written as

$$\frac{\partial B(\zeta)}{\partial y_i} = \frac{B_i}{(\zeta - y_i)^2} - \left[\frac{B_i}{\zeta - y_i}, B(\zeta)\right]. \tag{3.3}$$

Let us now introduce the tau-function of the Schlesinger system. We follow M. Jimbo, T. Miwa, and K. Ueno [20].

For any solution of the Schlesinger system, the 1-form

$$\omega = \sum_{i=1}^{k} \left(\sum_{j \neq i} \frac{\operatorname{tr}(B_i B_j)}{y_i - y_j} \right) dy_i$$

is closed. Locally, there exists a function τ with $d\log(\tau) = \omega$.

Using Schlesinger system, one easily computes

$$\frac{\partial^2 \log(\tau)}{\partial y_i \partial y_j} = \frac{\operatorname{tr}(B_i B_j)}{(y_i - y_j)^2}, \qquad \frac{\partial^2 \log(\tau)}{\partial y_i^2} = -\sum_{j \neq i} \frac{\operatorname{tr}(B_i B_j)}{(y_i - y_j)^2}. \tag{3.4}$$

Our goal is to show how (3.2) and (3.4) appear as limits of their discrete analogs.

3.2. Return now to the setting of Section 2.4. Let $\mathcal{A}(z)$ be a d-connection on $(\mathcal{O}(-1))^m$ with simple zeroes at n distinct points $a_1, \dots, a_n$, simple poles at n distinct points $b_1, \dots, b_n$, and no other singularities. Assume also that $\mathcal{A}(\infty) = I$. As usual, we assume that no two singularities differ by an integer. We consider the action of $\mathbb{Z}^n$ by isomonodromy transformations that shift a_i and b_i simultaneously.

For every $(u; v) = (u_1, \dots, u_n; v_1, \dots, v_n)$ such that $u_i - a_i = v_i - b_i \in \mathbb{Z}$, denote the corresponding modification of $\mathcal{A}(z)$ by $\mathcal{A}_{u;v}(z)$. The matrix of $\mathcal{A}_{u;v}(z)$ is denoted by $A_{u;v}(z)$. Also, for every index $i = 1, \dots, n$, we choose a basis $w_{u;v|i}$ of $\ker(A_{u;v}(u_i))$, and a basis $w'_{u;v|i}$ in the image of $\lim_{z \to v_i} (z - v_i) A^t_{u;v}(z)$. The choices for different $(u; v)$ have to be compatible as described in Section 2.4.

Let us introduce the following notation. For $(u; v)$ as above, set

$$D_i(\tau(u; v)) = \frac{\tau(u + e_i; v + e_i)}{\tau(u; v)}, \qquad i = 1, \dots, n;$$

$$D^2_{i,j}(\tau(u; v)) = \frac{\tau(u + e_i + e_j; v + e_i + e_j) \cdot \tau(u; v)}{\tau(u + e_i; v + e_i) \cdot \tau(u + e_j; v + e_j)}, \qquad i, j = 1, \dots, n.$$

Theorem 3.1. *Assume that our data depend on the small parameter $\varepsilon \neq 0$ so that*

$$a_i = \alpha_i + \frac{y_i}{\varepsilon}, \qquad b_i = \beta_i + \frac{y_i}{\varepsilon}, \qquad \lim_{\varepsilon \to 0} \frac{w_{a;b|i}(\varepsilon) \cdot (w'_{a;b|i}(\varepsilon))^t}{\langle w_{a;b|i}(\varepsilon), w'_{a;b|i}(\varepsilon) \rangle} = \frac{B_i}{\beta_i - \alpha_i}$$

for some matrices B_i.

Fix $(u; v)$ as above and $\zeta \in \mathbb{C} - \{y_1, \dots, y_n\}$. Then

$$A_{u;v}(\zeta \varepsilon^{-1}; \varepsilon) = I + \varepsilon \sum_{i=1}^{n} \frac{B_i}{\zeta - y_i} + o(\varepsilon); \tag{3.5}$$

$$\lim_{\varepsilon \to 0} \frac{A_{u+e_i;v+e_i}(\zeta \varepsilon^{-1}; \varepsilon) - A_{u;v}(\zeta \varepsilon^{-1}; \varepsilon)}{\varepsilon^2} = \frac{B_i}{(\zeta - y_i)^2} - \sum_{j \neq i} \frac{[B_i, B_j]}{(\zeta - y_i)(\zeta - y_j)}; \tag{3.6}$$

$$\lim_{\varepsilon \to 0} \frac{D^2_{i,j}(\tau(u; v; \varepsilon)) - 1}{\varepsilon^2} = \begin{cases} \dfrac{\operatorname{tr}(B_i B_j)}{(y_i - y_j)^2}, & (i \neq j) \\[2ex] -\displaystyle\sum_{k \neq i} \frac{\operatorname{tr}(B_i B_k)}{(y_i - y_k)^2}, & (i = j). \end{cases} \tag{3.7}$$

The right-hand sides of (3.5), (3.6), and (3.7) correspond to (3.1), (3.3), and (3.4).

Remark. Note that Theorem 3.1 leads to the Schlesinger system with rank one matrices B_i. One can obtain the general case by a proper limiting procedure bringing several singularities together.

3.3. **Proof of Theorem 3.1.**

Lemma 3.2. *Let $w_{a;b|i}(\varepsilon)$ and $w'_{a;b|i}(\varepsilon)$ $(i = 1, \dots, n)$ be vector-valued functions of $\varepsilon \neq 0$. Suppose that they satisfy the limit relation above. For ε small enough, there exists unique $A_{a;b}(z;\varepsilon)$ of the kind we consider that corresponds to these data.*

Proof. Uniqueness of $A_{a;b}(z;\epsilon)$ is almost obvious, since a rational matrix is determined by its singularity data and asymptotic behavior at infinity.

Let us prove existence. Proceed by induction in n. Set

$$R_{a;b|i}(z;\varepsilon) = I + \frac{\beta_i - \alpha_i}{z - b_i} \cdot \frac{w_{a;b|i}(\varepsilon) \cdot (w'_{a;b|i}(\varepsilon))^t}{\langle w_{a;b|i}(\varepsilon), w'_{a;b|i}(\varepsilon)\rangle}. \tag{3.8}$$

By the hypotheses,

$$R_{a;b|i}(\zeta\varepsilon^{-1};\varepsilon) = I + \varepsilon \frac{B_i}{\zeta - y_i} + o(\varepsilon). \tag{3.9}$$

We then construct $A_{a;b}(z;\epsilon)$ as

$$A_{a;b}(z;\epsilon) = \widetilde{A}_{a;b}(z;\epsilon) \cdot R_{a;b|n}(z;\varepsilon), \tag{3.10}$$

where $\widetilde{A}_{a;b}(z;\epsilon)$ is a matrix-valued function such that $\widetilde{A}_{a;b}(\infty;\epsilon) = I$, $\widetilde{A}_{a;b}(z;\epsilon)$ has simple zeroes at $a_1, \dots, a_{n-1}$ and simple poles at $b_1, \dots, b_{n-1}$ (and no other singularities), and $\ker(\widetilde{A}_{a;b}(a_i;\epsilon))$ (resp. image of $\lim_{z\to b_i}(z - b_i)\widetilde{A}^t_{u;v}(z;\epsilon)$) is spanned by $R_{a;b|n}(a_i;\epsilon)w_{a;b|i}(\epsilon)$ (resp. $R^{-t}_{a;b|n}(b_i;\epsilon)w'_{a;b|i}(\epsilon)$). Such $\widetilde{A}$ exists by the induction hypothesis. □

Let us now prove Theorem 3.1. Clearly, (3.5) follows from (3.9), (3.10).

Let us prove (3.6). Without losing generality, we can assume $(u;v) = (a - e_i, b - e_i)$. Then $A_{u;v}(z;\epsilon) = R_{a;b|i}(z+1;\epsilon)A_{a;b}(z;\epsilon)R_{a;b|i}(z,\epsilon)^{-1}$, where $R_{a;b|i}$ is given by (3.8). We then have

$$\begin{aligned} & A_{u+e_i;v+e_i}(\zeta\varepsilon^{-1};\varepsilon) - A_{u;v}(\zeta\varepsilon^{-1};\varepsilon) \\ &= A_{a;b}(\zeta\epsilon^{-1};\epsilon) - R_{a;b|i}(\zeta\epsilon^{-1}+1;\epsilon)A_{a;b}(\zeta\epsilon^{-1};\epsilon)R_{a;b|i}(\zeta\epsilon^{-1};\epsilon)^{-1} \\ &= \Big[I - R_{a;b|i}(\zeta\epsilon^{-1}+1;\epsilon)R_{a;b|i}\zeta\epsilon^{-1};\epsilon)^{-1}\Big] \\ &- \Big[R_{a;b|i}(\zeta\epsilon^{-1}+1;\epsilon)(A_{a;b}(\zeta\epsilon^{-1};\epsilon) - I)R_{a;b|i}(\zeta\epsilon^{-1};\epsilon)^{-1} - (A_{a;b}(\zeta\epsilon^{-1};\epsilon) - I)\Big]. \end{aligned}$$

Using (3.5) and (3.8), we see that the first bracket divided by ϵ^2 (resp. the second bracket divided by ϵ^2) converges to the first (resp. second) term in the right-hand side of (3.6).

It remains to prove (3.7). By (2.4), we have

$$\begin{aligned} D_i(\tau(u;v;\epsilon)) &= \frac{\alpha_i - \beta_i}{\langle w_{u+e_i;v+e_i|i}(\epsilon), w'_{u+e_i;v+e_i|i}(\epsilon)\rangle}, \\ D^2_{i,j}(\tau(u;v;\epsilon)) &= \frac{\langle w_{u+e_i;v+e_i|i}(\epsilon), w'_{u+e_i;v+e_i|i}(\epsilon)\rangle}{\langle w_{u+e_i+e_j;v+e_i+e_j|i}(\epsilon), w'_{u+e_i+e_j;v+e_i+e_j|i}(\epsilon)\rangle}. \end{aligned}$$

Without loss of generality, we can assume that $(u; v) = (a - e_i - e_j; b - e_i - e_j)$. For $i \neq j$, we have

$$w_{a-e_j;b-e_j|i}(\epsilon) = R_{a;b|j}(a_i;\epsilon) w_{a;b|i}(\epsilon), \qquad w'_{a-e_j;b-e_j|i}(\epsilon) = R^{-t}_{a;b|j}(b_i;\epsilon) w'_{a;b|i}(\epsilon).$$

Then

$$D^2_{i,j}(\tau(u;v;\epsilon)) = \frac{\langle R_{a;b|j}(a_i;\epsilon) w_{a;b|i}(\epsilon), R^{-t}_{a;b|j}(b_i;\epsilon) w'_{a;b|i}(\epsilon)\rangle}{\langle w_{a;b|i}(\epsilon), w'_{a;b|i}(\epsilon)\rangle}.$$

The difference of the numerator and the denominator equals

$$\langle (R_{a;b|j}(a_i;\epsilon) - R_{a;b|j}(b_i;\epsilon)) w_{a;b|i}(\epsilon), R^{-t}_{a;b|j}(b_i;\epsilon) w'_{a;b|i}(\epsilon)\rangle.$$

By (3.8),

$$R_{a;b|j}(a_i;\epsilon) - R_{a;b|j}(b_i;\epsilon) = \epsilon^2(\beta_i - \alpha_i)\frac{B_j}{(y_i - y_j)^2} + o(\epsilon^2).$$

This implies the statement.

Finally, suppose $i = j$. Then (cf. (2.7))

$$\begin{aligned} w_{a-e_i;b-e_i|i}(\epsilon) &= R_{a;b|i}(a_i - 1;\epsilon) A^{-1}_{a;b}(a_i - 1;\epsilon) w_{a;b|i}(\epsilon), \\ w'_{a-e_i;b-e_i|i}(\epsilon) &= R^{-t}_{a;b|i}(b_i - 1;\epsilon) A^{t}_{a;b}(b_i - 1;\epsilon) w'_{a;b|i}(\epsilon). \end{aligned}$$

The statement now follows from the asymptotics

$$\begin{aligned} R_{a;b|i}(a_i - 1;\epsilon) A^{-1}_{a;b}(a_i - 1;\epsilon) - R_{a;b|i}(b_i - 1;\epsilon) A^{-1}_{a;b}(b_i - 1;\epsilon) \\ = -\epsilon^2(\beta_i - \alpha_i)\sum_{j\neq i}\frac{B_j}{(y_i - y_j)^2} + o(\epsilon^2). \end{aligned}$$

□

4. Discrete Painlevé equations

In some special cases, isomonodromy transformation gives rise to discrete Painlevé equations ([23, 5, 7, 3, 31]). In this section, we evaluate the τ-function for the two cases considered in [3].

4.1. Difference P_V and difference P_{VI}.

Suppose $\mathcal{L}$ is a rank 2 vector bundle on $\mathbb{P}^1$, and that the d-connection $\mathcal{A}$ has simple zeroes at a_1, a_2, simple poles at b_1, b_2, and no other singularities. Also, fix the 'formal type' of $\mathcal{A}$ at infinity: there exists a trivialization $\mathcal{R}(z) : \mathbb{C}^2 \to \mathcal{L}_z$ on the formal neighborhood of infinity such that the matrix of $\mathcal{A}$ with respect to $\mathcal{R}$ equals

$$\mathcal{R}(z+1)^{-1}\mathcal{A}(z)\mathcal{R}(z) = \begin{bmatrix} \rho_1(1 + \frac{d_1+b_1+b_2+1}{z}) & 0 \\ 0 & \rho_2(1 + \frac{d_2+b_1+b_2+1}{z}) \end{bmatrix},$$

for $\rho_1, \rho_2, d_1, d_2 \in \mathbb{C}$. (This choice of parameters is used to match the formulas of [3].) Finally, suppose that

$$d_1 + d_2 + a_1 + a_2 + b_1 + b_2 = 0.$$

This implies that $\deg(\mathcal{L}) = -2$.

Assuming the parameters are generic, the moduli space of such d-connections is a surface (of type $D_4^{(1)}$), see [3]. Let us introduce the coordinates on this surface.

For generic $(\mathcal{L},\mathcal{A})$, there exists an isomorphism $\mathcal{L}\widetilde{\to}(\mathcal{O}(-1))^2$ such that the matrix of $\mathcal{A}$ is of the form

$$A(z)=\begin{bmatrix} a_{11}(z) & O(z)\\ z-q & \rho_2 z^2+\rho_2 d_2 z+O(1)\end{bmatrix}\cdot\frac{1}{(z-b_1)(z-b_2)},$$

where $a_{11}(z)$ is of the form $a_{11}(z)=\rho_1 z^2+\rho_1 d_1 z+O(1)$. $A(z)$ is uniquely determined by q and $a_{11}(q)$; the other coefficients can be found using the singularity structure of $A(z)$. We take the (rational) coordinates on the moduli space to be q and

$$p=\frac{a_{11}(q)}{(q-a_2)(q-b_2)}.$$

Consider the isomonodromy deformation that shifts $a_1\mapsto a_1-1$, $b_1\mapsto b_1-1$. According to our choice of parameters, it also shifts $d_1\mapsto d_1+1$, $d_2\mapsto d_2+1$, because the formal type at the infinity does not change.

Proposition 4.1 ([3, Theorem B])**.** *The transformed coordinates q', p' are related to p and q by the following equations (difference P_V):*

$$\begin{cases} q'+q=a_2+b_2+\dfrac{\rho_1(d_1+a_2+b_2)}{p-\rho_1}+\dfrac{\rho_2(d_2+a_2+b_2+1)}{p-\rho_2},\\ p'p=\dfrac{(q'-a_1+1)(q'-b_1+1)}{(q'-a_2)(q'-b_2)}\cdot\rho_1\rho_2.\end{cases}$$

□

Now consider d-connections with different singularity structure. Namely, suppose $\mathcal{A}$ has simple zeroes at a_1, a_2, a_3, simple poles at b_1, b_2, b_3, and no other singularities. Assume that in the formal neighborhood of infinity, there exists a trivialization $\mathcal{R}(z):\mathbb{C}^2\to\mathcal{L}_z$ such that the matrix of $\mathcal{A}$ with respect to $\mathcal{R}$ equals

$$\mathcal{R}(z+1)^{-1}\mathcal{A}(z)\mathcal{R}(z)=\begin{bmatrix}1+\frac{d_1+b_1+b_2+b_3+1}{z} & 0\\ 0 & 1+\frac{d_2+b_1+b_2+b_3+1}{z}\end{bmatrix},$$

for $d_1,d_2\in\mathbb{C}$. Finally, suppose that

$$d_1+d_2+a_1+a_2+a_3+b_1+b_2+b_3=0.$$

This implies that $\deg(\mathcal{L})=-2$.

For generic $(\mathcal{L},\mathcal{A})$, there exists an isomorphism $\mathcal{L}\widetilde{\to}(\mathcal{O}(-1))^2$ such that the matrix of $\mathcal{A}$ is of the form

$$A(z)=\begin{bmatrix} a_{11}(z) & O(z)\\ z-q & z^3+d_2 z^2+O(z)\end{bmatrix}\cdot\frac{1}{(z-b_1)(z-b_2)(z-b_3)},$$

where $a_{11}(z)$ is of the form $z^3+d_1z^2+O(z)$. Then $A(z)$ is determined by q and $a_{11}(q)$. It is more convenient to work in coordinates q and

$$r=\frac{(q-a_2)(q-a_3)(q-b_2)(q-b_3)}{a_{11}(q)}-q.$$

As above, consider the isomonodromy deformation that shifts $a_1\mapsto a_1-1$, $b_1\mapsto b_1-1$; it also shifts $d_1\mapsto d_1+1$, $d_2\mapsto d_2+1$.

Proposition 4.2 ([3, Theorem F]). *The transformed coordinates q', p' are related to p and q by the following equations (difference P_{VI}):*

$$\begin{cases} (q+r)(q'+r) = \dfrac{(r+a_2)(r+a_3)(r+b_2)(r+b_3)}{(r+1-a_1-b_1-d_1)(r-a_1-b_1-d_2)}\,, \\[2ex] (q'+r)(q'+r') = \dfrac{(q'-a_2)(q'-a_3)(q'-b_2)(q'-b_3)}{(q'-(a_1-1))(q'-(b_1-1))}\,. \end{cases}$$

□

Remark. These equations previously appeared in [17] as the asymmetric dPIV equation; see also references therein. The equivalence of [3, Theorem F] and the equations is explained in the introduction to [3].

4.2. **Tau-functions.** Following the recipe of Section 2.4, we can write the second (logarithmic difference) derivative of the tau-function in the direction of the above isomonodromy transformations. The computations are somewhat tedious, but the answer is remarkably simple:

Theorem 4.3. *In the settings of Proposition 4.1,*

$$\begin{aligned} D^2\tau = \frac{\tau''\cdot\tau}{(\tau')^2} &= \frac{(p'-\rho_1)(\rho_1(q'-a_1+1)(q'-b_1+1)-p'(q'-a_2)(q'-b_2))}{\rho_1(a_2-a_1+1)(b_2-b_1+1)p'} \\ &= \frac{(p'-\rho_1)(p-\rho_2)(q'-a_2)(q'-b_2)}{\rho_1\rho_2(a_2-a_1+1)(b_2-b_1+1)}\,. \end{aligned}$$

Here τ' and τ'' correspond to shifts $(a_1,b_1)\mapsto(a_1-1,b_1-1)$ and $(a_1,b_1)\mapsto(a_1-2,b_1-2)$, respectively. □

Theorem 4.4. *In the settings of Proposition 4.2,*

$$\begin{aligned} D^2\tau = \frac{\tau''\cdot\tau}{(\tau')^2} = {}& \frac{1}{(a_1-a_2-1)(a_1-a_3-1)(b_1-b_2-1)(b_1-b_3-1)} \\ &\times \frac{r'-a_1-b_1-d_2+1}{q'+r'}\cdot\big((q'-a_2)(q'-a_3)(q'-b_2)(q'-b_3) \\ &-(q'-a_1+1)(q'-b_1+1)(q'+d_1+a_1+b_1-1)(q'+r')\big) \\ ={}& \frac{(r'-a_1-b_1-d_2+1)(r-a_1-b_1-d_1+1)(q'-a_1+1)(q'-b_1+1)}{(a_1-a_2-1)(a_1-a_3-1)(b_1-b_2-1)(b_1-b_3-1)}\,. \end{aligned}$$

Here τ' and τ'' correspond to shifts $(a_1,b_1)\mapsto(a_1-1,b_1-1)$ and $(a_1,b_1)\mapsto(a_1-2,b_1-2)$, respectively. □

Remark. Propositions 4.1, 4.2, and Theorems 4.3, 4.4 remain valid in various degenerate situations. For example, zero at $z=a_1$ and pole at $z=b_1$ can coalesce, giving a singularity of the type considered in Section 2.5. This degeneration is used in Section 6.

5. Gap probabilities

The goal of this section is to show that tau-functions naturally arise as the gap probabilities in the discrete probabilistic models of random matrix type.

5.1. Fix a finite set $\mathfrak{X} \subset \mathbb{C}$ (the phase space), and two families of weight functions $\omega_{1,1}, \dots, \omega_{1,p}, \omega_{2,1}, \dots, \omega_{2,q}$ defined on $\mathfrak{X}$. Assume the weight functions have no zeroes on $\mathfrak{X}$. Also, fix two multi-indices $\mathbf{n} = (n_1, \dots, n_p), \mathbf{m} = (m_1, \dots, m_q)$ such that

$$N = \sum_{i=1}^{p} n_i = \sum_{i=1}^{q} m_i.$$

Set

$$F(x_1, \dots, x_N) = \det[\phi_i(x_j)]_{i,j=1}^{N} \det[\psi_i(x_j)]_{i,j=1}^{N},$$

where

$$\{\phi_i(x) \mid i = 1, \dots, N\} = \{\omega_{1,i}(x)x^j \mid i = 1, \dots, p, j = 0, \dots, n_i - 1\},$$
$$\{\psi_i(x) \mid i = 1, \dots, N\} = \{\omega_{2,i}(x)x^j \mid i = 1, \dots, q, j = 0, \dots, m_i - 1\}.$$

We always make the following basic assumption:

$$Z = \sum_{x_1, \dots, x_N \in \mathfrak{X}} F(x_1, \dots, x_N) \neq 0. \tag{5.1}$$

Remark. Let $\mathfrak{F}$ (resp. $\mathfrak{G}$) be the subspace of $\ell^2(\mathfrak{X})$ spanned by ϕ_i's (resp. ψ_i's). Then (5.1) is equivalent to $\dim(\mathfrak{F}) = \dim(\mathfrak{G}) = N$ and $\mathfrak{F} \cap \mathfrak{G}^\perp = \{0\}$.

Lemma 5.1. *Let $K(x,y)$ be the matrix of the projection in $\ell^2(\mathfrak{X})$ onto $\mathfrak{F}$ parallel to $\mathfrak{G}^\perp$:*

$$K(x,y) = \sum_{i,j=1}^{N} M_{ij}\phi_i(x)\psi_j(y) \quad \textit{for} \quad M = \|\langle \phi_i, \psi_j \rangle\|_{i,j=1,\dots,N}^{-t}.$$

Then for any subset $\mathfrak{Y} \subset \mathfrak{X}$

$$\frac{1}{Z} \sum_{x_1, \dots, x_N \in \mathfrak{Y}} F(x_1, \dots, x_N) = \det\left((1-K)\big|_{\ell^2(\mathfrak{X}-\mathfrak{Y})} \right).$$

The proof is a standard argument in the random matrix theory.

5.2. Consider on $\mathbb{P}^1$ the vector bundle

$$\mathcal{L}_\varnothing = \mathcal{O}(n_1 - 1) \oplus \dots \oplus \mathcal{O}(n_p - 1) \oplus \mathcal{O}(-m_1 - 1) \oplus \dots \oplus \mathcal{O}(-m_q - 1).$$

For any subset $\mathfrak{Y} \subset \mathfrak{X}$, define a modification $\mathcal{L}_\mathfrak{Y}$ of $\mathcal{L}_\varnothing$ by

(1) $\mathcal{L}_\mathfrak{Y}$ and $\mathcal{L}_\varnothing$ coincide on $\mathbb{P}^1 \setminus \mathfrak{Y}$;
(2) Near any $y \in \mathfrak{Y}$, sections of $\mathcal{L}_\mathfrak{Y}$ are rational sections

$$s = (s_{1,1}, \dots, s_{1,p}; s_{2,1}, \dots, s_{2,q})^t \in \mathcal{L}_\varnothing$$

such that $s_{1,i}$ is regular at y $(i = 1, \dots, p)$, $s_{2,i}$ has at most a first order pole at y $(i = 1, \dots, q)$, and

$$\operatorname{res}_y(s_{2,i}) = \omega_{2,i}(y) \cdot \sum_{j=1}^{p} \omega_{1,j}(y)s_{1,j}(y).$$

Note that $\deg(\mathcal{L}_\mathfrak{Y}) = \deg(\mathcal{L}_\varnothing) = -p - q$.

Proposition 5.2. *Under the assumption* (5.1), $\mathcal{L}_\mathfrak{X} \simeq (\mathcal{O}(-1))^{p+q}$.

Proof. This follows from a discrete version of [10, Theorem 3.1]. Since we do not need an explicit solution to the associated Riemann-Hilbert problem, we provide an independent argument.

Since $\deg(\mathcal{L}_{\mathfrak{X}}) = -p-q$, it suffices to show that $\mathcal{L}_{\mathfrak{X}}$ has no global sections. A global section of $\mathcal{L}_{\mathfrak{X}}$ is of the form

$$s = (s_{1,1}, \dots, s_{1,p}; s_{2,1}, \dots, s_{2,q})^t,$$

where $s_{1,i}$ is a polynomial in z of degree at most $n_i - 1$ ($i = 1, \dots, p$), and $s_{2,i}$ is given by

$$s_{2,i}(z) = \sum_{x \in \mathfrak{X}} \frac{\omega_{2,i}(x) \cdot \sum_{j=1}^{p} \omega_{1,j}(x) s_{1,j}(x)}{z-x}, \qquad (i = 1, \dots, q).$$

and satisfies the following condition:

(5.2) The order of zero of $s_{2,i}(z)$ at $z = \infty$ is at least $m_i + 1$.

Equivalently, (5.2) means that for any polynomial $p(z)$ of degree $m_i - 1$ or less,

$$\operatorname*{res}_{z=\infty} s_{2,i}(z)p(z) = 0.$$

Evaluating the residue as the sum over finite poles, we obtain

$$\sum_{x\in\mathfrak{X}} p(x)\omega_{2,i}(x) \cdot \sum_{j=1}^{p} \omega_{1,j}(x) s_{1,j}(x) = 0, \qquad (i = 1, \dots, p; \quad \deg(p) \le m_i - 1).$$

Equivalently, $\sum_{j=1}^{p} \omega_{1,j}(x)s_{1,j}(x)$ belongs to $\mathfrak{G}^{\perp} \cap \mathfrak{F}$, which is trivial by our assumption. □

Similarly to Section 2.5, we introduce at every point $x \in \mathfrak{X}$ a vector w_x and a functional $w'_x + w''_x(z-x)$:

$$\begin{aligned} w_x &= (0, \dots, 0; \omega_{2,1}(x), \dots, \omega_{2,q}(x))^t \\ w'_x &= (\omega_{1,1}(x), \dots, \omega_{1,p}(x); 0, \dots, 0)^t \\ w''_x &= \left(0, \dots, 0; \tfrac{1}{\omega_{2,1}(x)}, 0, \dots, 0\right)^t. \end{aligned} \tag{5.3}$$

Remark. In what follows, w''_x is important only modulo $w_x^{\perp}$. In this sense, the definition of w''_x is symmetric:

$$\left(0, \dots, 0; \tfrac{1}{\omega_{2,1}(x)}, 0, \dots, 0\right)^t \equiv \dots \equiv \left(0, \dots, 0; 0, \dots, 0, \tfrac{1}{\omega_{2,q}(x)}\right)^t \mod w_x^{\perp}.$$

The modification $\mathcal{L}_{\mathfrak{Y}}$ (for $\mathfrak{Y} \subset \mathfrak{X}$) can be described in terms of these data as follows: the sections of $\mathcal{L}_{\mathfrak{Y}}$ near $y \in \mathfrak{Y}$ are sections $s \in \mathcal{L}_{\varnothing}$ with at most a first order pole such that $\operatorname{res}_y s \in \mathbb{C}w_y$ and $(w'_y + w''_y(z-y))s|_{z=y} = 0$. Note that $\langle w_y, w'_y \rangle = 0$, so the last condition makes sense.

Theorem 5.3. *For any $\mathfrak{Y} \subset \mathfrak{X}$, we have*

$$\frac{\tau(\mathcal{L}_{\mathfrak{Y}})}{\tau(\mathcal{L}_{\mathfrak{X}})} = \det\left(1 - K|_{\ell^2(\mathfrak{X}-\mathfrak{Y})}\right).$$

Here we identify $\operatorname{detR\Gamma}(\mathcal{L}_{\mathfrak{Y}}) = \operatorname{detR\Gamma}(\mathcal{L}_{\mathfrak{X}})$ *by means of* (w_x, w'_x, w''_x)*,* $x \in \mathfrak{X}\setminus\mathfrak{Y}$*, so we can view the left-hand side as a number, see below.*

Let us describe the identification $\det R\Gamma(\mathcal{L}_{\mathfrak{Y}}) = \det R\Gamma(\mathcal{L}_{\mathfrak{X}})$ explicitly. Let $\mathcal{L}_{\mathfrak{Y}}^{up}$ be a modification of $\mathcal{L}_{\mathfrak{Y}}$ on $\mathfrak{X} - \mathfrak{Y}$ whose sections near $x \in \mathfrak{X} - \mathfrak{Y}$ are of the form

$$s = (s_{1,1}, \dots, s_{1,p}; s_{2,1}, \dots, s_{2,q})^t \in \mathcal{L}_{\mathfrak{Y}},$$

where $s_{1,i}$ is regular at x ($i = 1, \dots, p$), $s_{2,i}$ has at most a first order pole at x ($i = 1, \dots, q$), and

$$\operatorname{res}_x s \sim w_x = (0, \dots, 0; \omega_{2,1}(x), \dots, \omega_{2,q}(x))^t.$$

Note that $\mathcal{L}_{\mathfrak{Y}}^{up}$ is also an upper modification of $\mathcal{L}_{\mathfrak{X}}$.

For every point $x \in \mathfrak{X} - \mathfrak{Y}$, consider two functionals on sections of $\mathcal{L}_{\mathfrak{Y}}^{up}$:

$$f_x(s) = (w_x' + w_x''(z - x))s|_{z=x},$$
$$g_x(s) = \operatorname{res}_x s/w_x.$$

Note that sections of $\mathcal{L}_{\mathfrak{X}}$ (resp. $\mathcal{L}_{\mathfrak{Y}}$) are exactly sections of $\mathcal{L}_{\mathfrak{Y}}^{up}$ on which f_x (resp. g_x) vanish for all $x \in \mathfrak{X} - \mathfrak{Y}$. In this way, we get identifications

$$f_{\mathfrak{X}-\mathfrak{Y}} = (f_x)_{x \in \mathfrak{X}-\mathfrak{Y}} : H^0(\mathbb{P}^{\mathbf{1}}, \mathcal{L}_{\mathfrak{Y}}^{up}/\mathcal{L}_{\mathfrak{X}}) \widetilde{\to} \mathbb{C}^{|\mathfrak{X}|-|\mathfrak{Y}|}$$
$$g_{\mathfrak{X}-\mathfrak{Y}} = (g_x)_{x \in \mathfrak{X}-\mathfrak{Y}} : H^0(\mathbb{P}^{\mathbf{1}}, \mathcal{L}_{\mathfrak{Y}}^{up}/\mathcal{L}_{\mathfrak{Y}}) \widetilde{\to} \mathbb{C}^{|\mathfrak{X}|-|\mathfrak{Y}|}.$$

This induces an isomorphism (see Example 1.12):

$$\det R\Gamma(\mathcal{L}_{\mathfrak{X}}) \otimes \det R\Gamma(\mathcal{L}_{\mathfrak{Y}})^{-1} = \det(H^0(\mathbb{P}^{\mathbf{1}}, \mathcal{L}_{\mathfrak{Y}}^{up}/\mathcal{L}_{\mathfrak{Y}})) \otimes \det(H^0(\mathbb{P}^{\mathbf{1}}, \mathcal{L}_{\mathfrak{Y}}^{up}/\mathcal{L}_{\mathfrak{X}}))^{-1} = \mathbb{C}.$$

In other words, the ratio $\tau(\mathcal{L}_{\mathfrak{Y}})/\tau(\mathcal{L}_{\mathfrak{X}})$ is the determinant of the composition

$$\mathbb{C}^{|\mathfrak{X}|-|\mathfrak{Y}|} \widetilde{\to} H^0(\mathbb{P}^{\mathbf{1}}, \mathcal{L}_{\mathfrak{Y}}^{up}/\mathcal{L}_{\mathfrak{X}}) \simeq H^0(\mathbb{P}^{\mathbf{1}}, \mathcal{L}_{\mathfrak{Y}}^{up}) \to H^0(\mathbb{P}^{\mathbf{1}}, \mathcal{L}_{\mathfrak{Y}}^{up}/\mathcal{L}_{\mathfrak{Y}}) \widetilde{\to} \mathbb{C}^{|\mathfrak{X}|-|\mathfrak{Y}|}.$$

Proof of Theorem 5.3. Define an embedding $\iota : H^0(\mathbb{P}^{\mathbf{1}}, \mathcal{L}_{\mathfrak{Y}}^{up}) \hookrightarrow \ell^2(\mathfrak{X})$ as follows: given

$$s = (s_{1,1}, \dots, s_{1,p}; s_{2,1}, \dots, s_{2,q}) \in H^0(\mathbb{P}^{\mathbf{1}}, \mathcal{L}_{\mathfrak{Y}}^{up}),$$

we set ($x \in \mathfrak{X}$)

$$\phi(x) = \sum_{i=1}^{p} s_{1,i}(x)\omega_{1,i}(x), \qquad \psi(x) = \operatorname{res}_x(s)/w_x, \qquad \iota(s) = \phi + \psi.$$

Note that $\phi \in \mathfrak{F}$, $\psi \in \mathfrak{G}^\perp$ (see proof of Proposition 5.2), so s is uniquely determined by $\iota(s)$.

The image $\iota(H^0(\mathbb{P}^{\mathbf{1}}, \mathcal{L}_{\mathfrak{Y}}^{up}))$ is the space of functions supported by $\mathfrak{X} - \mathfrak{Y}$. The functionals f_x and g_x can be written as $f_x(s) = (\iota(s))(x)$, $g_x(s) = (\pi \circ \iota(s))(x)$, where $\pi : \ell^2(\mathfrak{X}) \to \ell^2(\mathfrak{X})$ is the projection onto $\mathfrak{G}^\perp$ parallel to $\mathfrak{F}$ (that is to say, $\pi(\iota(s)) = \psi(x)$). Thus, the ratio $\tau(\mathcal{L}_{\mathfrak{Y}})/\tau(\mathcal{L}_{\mathfrak{X}})$ equals the determinant of the composition

$$\ell^2(\mathfrak{X} - \mathfrak{Y}) \hookrightarrow \ell^2(\mathfrak{X}) \xrightarrow{\pi} \ell^2(\mathfrak{X}) \to \ell^2(\mathfrak{X} - \mathfrak{Y}).$$

□

Remark 5.4. The entries of the matrix $1 - K$ can be interpreted as ratios of suitable τ-functions. Namely, fix $x, y \in \mathfrak{X}$, and consider the modification $\mathcal{L}_{\mathfrak{X}}^{\uparrow x, \downarrow y}$ of $\mathcal{L}_{\mathfrak{X}}$ at x and y such that for $x \neq y$

- Sections of $\mathcal{L}_{\mathfrak{X}}^{\uparrow x,\downarrow y}$ near x are of the form
$$s = (s_{1,1},\dots,s_{1,p};s_{2,1},\dots,s_{2,q})^t \in \mathcal{L}_{\mathfrak{X}},$$
where $s_{1,i}$ is regular at x ($i = 1,\dots,p$), $s_{2,i}$ has at most a first order pole at x ($i = 1,\dots,q$), and
$$\operatorname{res}_x s \sim w_x = (0,\dots,0;\omega_{2,1}(x),\dots,\omega_{2,q}(x))^t;$$
- Sections of $\mathcal{L}_{\mathfrak{X}}^{\uparrow x,\downarrow y}$ near y are of the form
$$s = (s_{1,1},\dots,s_{1,p};s_{2,1},\dots,s_{2,q})^t \in \mathcal{L}_{\mathfrak{X}},$$
where $s_{1,i}$ is regular at y ($i = 1,\dots,p$), $s_{2,i}$ is regular at y ($i = 1,\dots,q$), and
$$\sum_{i=1}^{p}\omega_{1,i}(y)s_{1,i}(y) = 0.$$

Thus if $x \neq y$, $\mathcal{L}_{\mathfrak{X}}^{\uparrow x,\downarrow y}$ is an elementary upper modification of $\mathcal{L}$ at x and its elementary lower modification at y. If $x = y$, we set $\mathcal{L}_{\mathfrak{X}}^{\uparrow x,\downarrow y} = \mathcal{L}_{\mathfrak{X}-\{x\}}$.

Taking $\mathfrak{Y} = \mathfrak{X} - \{x\}$, we see that $\mathcal{L}_{\mathfrak{Y}}^{up}$ is an upper modification of both $\mathcal{L}_{\mathfrak{X}}$ and $\mathcal{L}_{\mathfrak{X}}^{\uparrow x,\downarrow y}$. Under ι, $H^0(\mathbb{P}^1,\mathcal{L}_{\mathfrak{Y}}^{up})$ goes to the (one-dimensional) space of functions supported by $\{x\}$. The (x,y)-entry of $\pi = 1 - K$ is the ratio of functionals g_y/f_x on this one-dimensional space, which equals $\tau(\mathcal{L}_{\mathfrak{X}}^{\uparrow x,\downarrow y})/\tau(\mathcal{L}_{\mathfrak{X}})$. This statement can be viewed as a discrete analog of [10, Theorem 4.3].

From the point of view of Section 2.6, Theorem 5.3 is a Hirota type determinantal identity similar to Proposition 2.4.

5.3. Suppose now that there are rational functions
$$\varpi_{1,1}(z),\dots,\varpi_{1,p}(z);\varpi_{2,1}(z),\dots,\varpi_{2,q}(z)$$
such that for any $x \in \mathfrak{X}$ such that $x+1 \in \mathfrak{X}$, we have
$$\frac{\omega_{1,i}(x+1)}{\omega_{1,i}(x)} = \varpi_{1,i}(x), \qquad \frac{\omega_{2,i}(x+1)}{\omega_{2,i}(x)} = \varpi_{2,i}(x).$$
In particular, $\varpi_{1,i}$ and $\varpi_{2,i}$ are regular nonzero at x.

On
$$\mathcal{L}_{\varnothing} = \mathcal{O}(n_1-1)\oplus\cdots\oplus\mathcal{O}(n_p-1)\oplus\mathcal{O}(-m_1-1)\oplus\cdots\oplus\mathcal{O}(-m_q-1),$$
consider the d-connection
$$\mathcal{A}(z) = \operatorname{diag}\left(\frac{1}{\varpi_{1,1}(z)},\dots,\frac{1}{\varpi_{1,p}(z)},\varpi_{2,1}(z),\dots,\varpi_{2,q}(z)\right).$$

For $a,b \in \mathbb{C}$ such that $b-a \in \mathbb{Z}_{>0}$, we call the set $[a,b]_{\mathbb{Z}} = \{a,a+1,\dots,b\}$ the *integral segment* with endpoints a and b. Suppose $\mathbf{a} = (a_1,\dots,a_n)$, $\mathbf{b} = (b_1,\dots,b_n)$ are such that $[a_i,b_i]_{\mathbb{Z}}$, $i = 1,\dots,n$, are non-intersecting integral segments contained in $\mathfrak{X}$. Set
$$D(\mathbf{a},\mathbf{b}) = \frac{1}{Z}\sum_{x_1,\dots,x_N\in\mathfrak{X}-\bigcup_i [a_i,b_i]_{\mathbb{Z}}} F(x_1,\dots,x_N) = \det\left(1 - K\big|_{\ell^2(\bigcup_i [a_i,b_i]_{\mathbb{Z}})}\right)$$
(see Lemma 5.1 for the last equality).

Set $\mathfrak{Y} = \mathfrak{X} - \bigcup_i [a_i,b_i]_{\mathbb{Z}}$, and consider $\mathcal{A}(z)$ as a d-connection on $\mathcal{L}_{\mathfrak{Y}}$. For every i such that $b_i + 1 \in \mathfrak{Y}$, the connection $\mathcal{A}(z)$ has at $z = b_i$ a singularity of type described in Section 2.5. The corresponding modification of $\mathcal{L}_{\mathfrak{Y}}$ is exactly

$\mathcal{L}_{\mathfrak{Y}-\{b_i+1\}}$. Following Section 2.5, we need to choose at $z = b_i + 1$ a vector w and a functional $w' + (z - b_i - 1)w''$. It is natural to set $w = w_{b_i+1}$, $w' = w'_{b_i+1}$, $w'' = w''_{b_i+1}$, with the right-hand sides defined by (5.3). Moreover, it is explained in Section 2.5 that a choice of $(w, w', w'' \mod w^{\perp})$ at $z = b_i + 1$ determines the corresponding choice at $z = b_i + 2$. Our definition of $\mathcal{A}(z)$ is such that the new triple is exactly w_{b_i+2}, w'_{b_i+2}, w''_{b_i+2}. Here we assume that $b_i + 2 \in \mathfrak{Y}$.

Similar arguments apply to a_i. These observations imply the following statement.

Theorem 5.5. *Suppose that*

$$\mathcal{L}_{\mathfrak{Y}} \simeq (\mathcal{O}(-1))^{p+q}, \tag{5.4}$$

and let $A(z)$ be the matrix of $\mathcal{A}(z)$ corresponding to a choice of isomorphism (5.4). *Then $D(\mathbf{a}, \mathbf{b})$ is a τ-function of $A(z)$ in the sense that its second difference logarithmic derivatives*

$$\frac{D(\mathbf{a}, \mathbf{b} + e_i + e_j) \cdot D(\mathbf{a}, \mathbf{b})}{D(\mathbf{a}, \mathbf{b} + e_i) \cdot D(\mathbf{a}, \mathbf{b} + e_j)}$$

can be computed using the recipe of Section 2.5. The same statement holds true for derivatives with respect to $\mathbf{a}$ *and for the mixed derivatives.* □

6. Example: Hahn orthogonal polynomial ensemble

In the notation of the previous section, take $\mathfrak{X} = \{0, \dots, M\}$, $p = q = 1$, $m_1 = n_1 = N$. Set

$$\omega_1(x) = \frac{\Gamma(\alpha + x + 1)}{x!}, \qquad \omega_2(x) = \frac{\Gamma(\beta + M - x + 1)}{(M - x)!}.$$

This corresponds to

$$F(x_1, \dots, x_N) = \prod_{1 \le i < j \le N} (x_i - x_j)^2 \prod_{i=1}^{N} \omega_1(x_i)\omega_2(x_i).$$

Note that if α and β are such that $\omega_1(x)\omega_2(x) > 0$ for $x \in \mathfrak{X}$, $\omega_1\omega_2$ is the weight function for the classical Hahn orthogonal polynomials.

Set

$$D(s) = \frac{1}{Z} \sum_{x_1, \dots, x_N \le s} F(x_1, \dots, x_N).$$

Theorem 6.1. *For generic α and β, there exist sequences (q_s, r_s), where $s = N - 1, \dots, M$, that satisfy the difference P_{VI} of Proposition 4.2 with*

$$a_1 = s, \quad a_2 = -1, \quad a_3 = M,$$
$$b_1 = s, \quad b_2 = -\alpha - 1, \quad b_3 = \beta + M,$$
$$d_1 + b_1 + b_2 + b_3 = -\alpha - N, \quad d_2 + b_1 + b_2 + b_3 = \beta + N,$$

such that the second derivative of $D(s)$ is given by Theorem 4.4. Here

$$q = q_s, \quad q' = q_{s-1}, \qquad r = r_s, \quad r' = r_{s-1},$$
$$\tau = D(s), \quad \tau' = D(s-1), \quad \tau'' = D(s-2).$$

Remarks. 1. We assume that α and β are generic so that all bundles involved are isomorphic to $(\mathcal{O}(-1))^2$. However, one can view α and β as parameters and the statement of Theorem 6.1 as an identity between rational functions. If α and β are such that $\omega_1(x)\omega_2(x) > 0$ on $\mathfrak{X}$, all bundles are isomorphic to $(\mathcal{O}(-1))^2$ by Proposition 5.2.

2. The initial conditions for the recurrences (that is, p_{N-1}, q_{N-1}, $D(N-1)$, and $D(N)$) can be explicitly evaluated using the algorithm of [7, Section 6].

3. Consider the limit $M \to \infty$. If we scale the lattice $\mathfrak{X}$ by M^{-1}, the Hahn orthogonal polynomials converge to the Jacobi orthogonal polynomials on $[0, 1]$ (with same parameters α, β), and $D(s)$ converges to the corresponding quantity for the Jacobi polynomial ensemble. At the same time, the d-connections become ordinary connections and discrete isomonodromy transformations converge to the continuous isomonodromy deformations, as in Section 3; see also [6, Section 5]. In the one-interval case $\mathfrak{Y} = \{s+1, \ldots, M\}$, this corresponds to the degeneration of $dPVI$ (from Theorem 6.1) into classical PVI. This degeneration is described in [3, Section 6.4]. In the continuous setting, a description of the relation between isomonodromy transformation and the Jacobi polynomial ensemble, including the PVI case, can be found in [8, Section 8.1].

Proof. We set
$$\varpi_1(z) = \frac{\alpha + z + 1}{z + 1}, \qquad \varpi_2(z) = \frac{z - M}{z - M - \beta}.$$
Thus the matrix $A(z) = \operatorname{diag}(\varpi_1(z)^{-1}, \varpi_2(z))$ has simple zeroes at -1 and $-M$, simple poles at $-\alpha - 1$, $\beta + M$, and it behaves at infinity as $1 + \operatorname{diag}(-\alpha, \beta)/z + O(z^{-2})$. If we now consider the corresponding d-connection $\mathcal{A}$ on $\mathcal{L}_{\mathfrak{Y}}$ for $\mathfrak{Y} = \{s+1, \ldots, M\}$, we see that it has simple zeroes at -1, $-M$, simple poles at $-\alpha-1$, $\beta + M$, and that at $z = s$, its singularity is of the type considered in Section 2.5. Finally, on the formal neighborhood of infinity, there exists a trivialization $\mathcal{R}(z) : \mathbb{C}^2 \to \mathcal{L}_z$ such that the matrix of $\mathcal{A}$ with respect to $\mathcal{R}$ equals
$$\mathcal{R}(z+1)^{-1}\mathcal{A}(z)\mathcal{R}(z) = \begin{bmatrix} 1 + \frac{-\alpha-N+1}{z} & 0 \\ 0 & 1 + \frac{\beta+N+1}{z} \end{bmatrix}.$$
Proposition 4.2, Theorem 4.4, and Theorem 5.3 conclude the proof. $\square$

References

[1] M. Adler and P. van Moerbeke. Hermitian, symmetric and symplectic random ensembles: PDEs for the distribution of the spectrum. *Ann. of Math. (2)*, 153(1):149–189, 2001.

[2] M. Adler and P. van Moerbeke. Recursion relations for unitary integrals, combinatorics and the Toeplitz lattice. *Comm. Math. Phys.*, 237(3):397–440, 2003. `Math-ph/0201063`.

[3] D. Arinkin and A. Borodin. Moduli spaces of d-connections and difference Painleve equations. em Duke Math. J., 134(3):515–556, 2006. `Math/0411584`.

[4] J. Baik. Riemann-Hilbert problems for last passage percolation. In *Recent developments in integrable systems and Riemann-Hilbert problems (Birmingham, AL, 2000)*, volume 326 of *Contemp. Math.*, pages 1–21. Amer. Math. Soc., Providence, RI, 2003.

[5] A. Borodin. Discrete gap probabilities and discrete Painlevé equations. *Duke Math. J.*, 117(3):489–542, 2003. `Math-ph/0111008`.

[6] A. Borodin. Isomonodromy transformations of linear systems of difference equations. Ann. Math., 160 (2004), 1141–1182. `Math.CA/0209144`.

[7] A. Borodin and D. Boyarchenko. Distribution of the first particle in discrete orthogonal polynomial ensembles. *Comm. Math. Phys.*, 234(2):287–338, 2003. `Math-ph/0204001`.

[8] A. Borodin, and P. Deift. Fredholm determinants, Jimbo-Miwa-Ueno τ-functions, and representation theory.*Comm. Pure Appl. Math.*, 55(9):1160–1230, 2002. `Math-ph/0111007`.

[9] A. Borodin and G. Olshanski. Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes. *Ann. of Math. (2)*, 161(3):1319–1422, 2005. `Math/0109194`.

[10] E. Daems, A. B. J. Kuijlaars. Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions. `Math.CA/0511470`.

[11] B. Dubrovin. Geometry of 2D topological field theories. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, Lecture Notes in Math., 1620:120–348, Springer, Berlin, 1996.

[12] P. J. Forrester and N. S. Witte. Application of the τ-function theory of Painlevé equations to random matrices: PIV, PII and the GUE. *Comm. Math. Phys.*, 219(2):357–398, 2001. `Math-ph/0103025`.

[13] P. J. Forrester and N. S. Witte. Application of the τ-function theory of Painlevé equations to random matrices: $\mathrm{P_V}$, $\mathrm{P_{III}}$, the LUE, JUE, and CUE. *Comm. Pure Appl. Math.*, 55(6):679–727, 2002. `Math-ph/0201051`.

[14] P. J. Forrester and N. S. Witte. Discrete Painlevé equations and random matrix averages. *Nonlinearity*, 16(6):1919–1944, 2003. `Math-ph/0304020`.

[15] P. J. Forrester and N. S. Witte. Discrete Painlevé equations, orthogonal polynomials on the unit circle, and N-recurrences for averages over $U(N)$—$\mathrm{P_{III'}}$ and $\mathrm{P_V}$ τ-functions. *Int. Math. Res. Not.*, 4:160–183, 2004. `Math-ph/0305029`.

[16] P. J. Forrester and N. S. Witte. Application of the τ-function theory of Painlevé equations to random matrices: $\mathrm{P_{VI}}$, the JUE, CyUE, cJUE and scaled limits. *Nagoya Math. J.*, 174:29–114, 2004. `Math-ph/0308036`.

[17] B. Grammaticos, A. Ramani, and Y. Ohta. A unified description of the asymmetric q-$\mathrm{P_V}$ and d-$\mathrm{P_{IV}}$ equations and their Schlesinger transformations. *J. Nonlinear Math. Phys.* 10(2): 215–228, 2003.

[18] J. Harnad and A. R. Its. Integrable Fredholm operators and dual isomonodromic deformations. *Comm. Math. Phys.*, 226(3):497-530, 2002.

[19] M. Jimbo, T. Miwa, T. Môri, and M. Sato. Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent. *Physica D*, 1:80-158, 1980.

[20] M. Jimbo, T. Miwa, and K. Ueno. Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I. *Physica D*, 2:306–352, 1981.

[21] M. Jimbo and T. Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II. *Physica D*, 2(3): 407–448, 1981.

[22] M. Jimbo and T. Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients III. *Physica D*, 4(1):26–46, 1981/82.

[23] M. Jimbo and H. Sakai. A q-analog of the sixth Painlevé equation. *Lett. Math. Phys.*, 38(2):145–154, 1996.

[24] K. Johansson Non-intersecting paths, random tilings and random matrices. *Probab. Theory Related Fields*, 123(2):225–280, 2002. `Math/0011250`.

[25] F. F. Knudsen and D. Mumford The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div". *Math. Scand.*, 39(1):19–55, 1976.

[26] I. Krichever. Analytic theory of difference equations with rational and elliptic coefficients and the Riemann-Hilbert problem. *Uspekhi Mat. Nauk*, 59(6):111–150, 2004; translation in *Russian Math. Surveys*, 59(6):1117–1154, 2004. `Math-ph/0407018`.

[27] Y. Laszlo and C. Sorger. The line bundles on the moduli of parabolic G-bundles over curves and their sections. *Ann. Sci. cole Norm. Sup.*, 30(4):499-525, 1997.

[28] M. L. Mehta. A nonlinear differential equation and a Fredholm determinant *J. Physique I*, 2(9):1721–1729, 1992.

[29] J. Palmer. Deformation analysis of matrix models *Physica D*, 78:166–185, 1994.

[30] H. Sakai. Rational surfaces associated with affine root systems and geometry of the Painlevé equations. *Comm. Math. Phys.*, 220(1):165–229, 2001.

[31] H. Sakai. Lax form of the q-Painlevé equation associated with the $A_2^{(1)}$ surface. *Jour. Phys. A*, 39(39):12203–12210, 2006.

[32] C. A. Tracy and H. Widom. Fredholm determinants, differential equations and matrix models. *Comm. Math. Phys.*, 163:33–72, 1994. `Hep-th/9306042`.